SOUND SYNTHESIS MODEL BASED ON THE SIMULATION OF A GAUSSIAN BOUNCING WAVE PACKET

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ABSTRACT
This article describes two synthesis approaches for the sonification of the quantum momenta distribution of a 1-D Gaussian bouncing wave packet model. These two sound synthesis systems were implemented in the Pure Data environment and both were tested comparatively. One system is based on a wavetable synthesis approach and the other is based on additive synthesis concepts. Both implementations are analyzed from a quantum mechanical point of view to ensure their correct functioning, and also their performance is compared based on different criteria and the gamma of produced sounds are studied under a perceptual basis. The sound capabilities of the model and the implications of the results are also discussed.

1. INTRODUCTION
The present article describes a multidisciplinary research consisting in designing a digital sound synthesis algorithm based upon the sonification of the bouncing wave packet quantum mechanical model. This is a bounded states problem well studied in physics that has its time dependent solutions already solved [1] [7]. Our problem resides in making an auditory display of this situation and in order to achieve this we need to find mechanisms for making a proper sonification and search for its applications as a system for producing, or synthesizing, sounds. This sound system is intended to provide a new sound imagery to the music composer based on the natural knowledge of quantum physics. Also it can serve as a pedagogical tool, by taking in consideration sonification as a reinforcement for visualization and contributing to attract and retain more students to the area of physics. In addition, the sonification represents scientific information as sound, facilitating the searching of patterns that would otherwise be hard to see.

Two sound synthesis models were constructed differing in their approach to sonification, in order to dispose of a greater range of applicability. We comparatively test their performances and give insights on further applications of the proposed sound synthesis systems.

This article is structured as follows: first the background for our work is presented including a discussion of related work and the most important aspects of the theory behind the Gaussian bouncing wave packet model. We then provide the details of the two sound synthesis approaches including the implementation procedures. Following this, we conduct an analysis of the implemented methods based on their performance, physical analysis and perceptual properties. Finally, we present the main conclusions and contributions of our work.

2. BACKGROUND
2.1. Related work
In 1990, Bain [2] explored some musical possibilities of the Schrödinger equation, the basic equation of quantum mechanics that allows for the calculation of a wavefunction given a set of initial conditions, including the case of the particle in a box. In his work, he considered different musical mappings, mostly based on graphs of the wave equation generated radially on the complex plane. These graphs are translated into the musical domain by the generation of pitches in the chromatic scale. As a result, the movement through pitch space represents the probability of where the particle is in a given energy level.

More recent advances in sound synthesis based on quantum physics are the ones made by Sturm [9], who derived a sound synthesis technique from the situation of quantum particles under the effect of an harmonic potential. Sturm constructed a metaphor between the data and the sound particles under the effect of an harmonic potential. Sturm constructed a metaphor between the data and the sound by matching each frequency in the sound to one particle in the system, and the frequency of that wave directly corresponding to the energy of that particle. The amplitude of that frequency is how far the particle is from the observer and the location of the particle relative to the observer determines the aural position of that frequency. In addition, the particles would be affected by external potentials thus affecting the overall sound.

Cádez and Kendall [3], constructed an audio synthesis technique based on sound particles with defined behavioral rules and the use of fuzzy logic. The model establishes that each sound particle in the system has certain fuzzy properties like frequency, intensity and charge, and that these properties are used as inputs to rule-based fuzzy logic inference system that controls the time evolution of the particles.

In the field of sonification, a sonification of the quantum spectra of baryons was performed by de Campo, Frauenberger and Höldrich [4] for classifying and explaining baryon
properties in the context of particle theory. Later, a sound-
ing model based upon the quantum harmonic oscillator 
was made by Saranti, Eckel and Pirr [8], showing applic-
ations in sound production and information storage.

2.2. The Gaussian bouncing wave packet

The behavior of certain 1-D quantum particle of definite 
mass and position can be mathematically represented by 
a wave function of space. The state of the particle can be 
understood as a combination of independent oscillatory 
modes that are permitted by the nature of the problem.

When such a combination of states concentrates in a cer-
tain point of space, the particle is called wave packet. The 
particle travels with a given velocity called group velocity,
as it represents the collective movement of the states com-
posing the particle. A bouncing wave packet is the situa-
tion that arises when we put the moving particle inside of 
a container of known dimensions with infinite tall walls. 

We are searching for the sonification of the model of a 
Gaussian-shaped bouncing wave packet, as it is relatively 
easy to describe and solve. The time-dependent solution 
of this problem for a Gaussian quantum wave packet of 
mass \( m \), momentum \( p_0 \), spreading \( \alpha \), initial center of pos-
tion \( x_0 \) inside of a one-dimensional container of length \( L \) 
is described by the wave function

\[
\psi(x,t) = C \sum_n \sin\left(\frac{n\pi x}{L}\right) \left[ \cos(n^2 E_0 t - \frac{n\pi x_0}{L}) A - \cos(n^2 E_0 t + \frac{n\pi x_0}{L}) B \right]. 
\]  

where

\[
A = e^{-\frac{\alpha^2}{2}(p_0 + n\frac{2\pi}{L})^2}, 
\]

\[
B = e^{-\frac{\alpha^2}{2}(p_0 - n\frac{2\pi}{L})^2}, 
\]

and \( C \) is a constant depending on \( \alpha \) and \( L \). We can identify 
the coefficients of the eigenstate base expansion

\[
\psi(x,t) = \sum_n C_n(t) \sin\left(\frac{n\pi x}{L}\right) 
\]

as

\[
C_n = C \left( \cos(n^2 E_0 t - \frac{n\pi x_0}{L}) A - \cos(n^2 E_0 t + \frac{n\pi x_0}{L}) B \right). 
\]

\[
C = \frac{\sqrt{2\pi\alpha}}{L} 
\]

We are now able to represent the wave equation of the 
Gaussian bouncing wave packet in the position space and 
to express the weight distribution of its states expansion. 

In figure 1, we can see the shape of a properly localized 
and normalized Gaussian wave packet inside an infinite 
potential well of length \( L \) at a time \( t \) close to 0.

When a bound-state wave packet is excited with an 
energy spectrum which is narrowly spread around a large 
central value of the quantum number \( n_0 \) that depends on 
the value of \( p_0 \), we can expand the individual energy eigen-
values \( E(n) \) about \( n_0 \), as: [7]

\[
E(n) \approx E(n_0) + E'(n_0)(n - n_0) + E''(n_0)(n - n_0)^2 + \ldots 
\]

where \( E'(n_0) = \frac{dE}{dn}(n_0) \) and so forth. Given this, the 
time dependence of each individual eigenstate is, to the 
second order (the other terms vanish in this problem):

\[
\exp(-iE_{\text{cl}}t) = \exp(-iE(n_0)t + (n - n_0)E'(n_0)t) 
+ \frac{1}{2} (n - n_0)^2 E''(n_0)t. 
\]

This equation can be expressed as

\[
\exp(-iE_{\text{cl}}t) = \exp(-i\omega_0 t - 2\pi i(n - n_0)t/T_{\text{cl}}) 
- (n - n_0)^2 t/T_{\text{rev}}), 
\]

where \( \omega_0 = E(n_0)/\hbar \) is a term that only contributes to 
a phase factor that can be cancelled by shifting the energy 
levels, and where the terms

\[
T_{\text{cl}} = \frac{2\pi\hbar}{E'(n_0)} \quad \text{and} \quad T_{\text{rev}} = \frac{2\pi\hbar}{E''(n_0)/2}, 
\]

represent important characteristic of the time scales of 
the system. In our situation, we recall Eqn.7 and obtain that

\[
E'(n_0) = 2n_0E_0 \quad \text{and} \quad E''(n_0) = 2E_0. 
\]
quantum revival in which the spreading reverses itself, the wave packet re-localizes, and the semi-classical periodicity is once again evident.

The revival time can be rewritten as

\[ T_{\text{rev}} = \frac{2\pi \hbar}{|E_0|} \]  

(12)

This time scale marks a whole lap in the time-evolution of the system, since for all \( t \)

\[ \psi(x, t + T_{\text{rev}}) = \sum_n a_n u_n(x) e^{-iE_n t} e^{-iE_n T_{\text{rev}}} \]

\[ = \sum_n a_n u_n(x) e^{-iE_n t} e^{-i2\pi n^2} \]  

(13)

The wave packet will also reform itself at \( t = T_{\text{rev}}/2 \) in the so-called \textit{mirror revival} [1], since

\[ \psi(L - x, t + T_{\text{rev}}) = \sum_n a_n u_n(L - x) e^{-iE_n t} e^{-iE_n T_{\text{rev}}} \]

\[ = \sum_n a_n u_n(x) e^{-iE_n t} (-1)^n e^{-i2\pi n^2} = -\psi(x, t). \]  

(14)

Here we used the parity property of the eigenfunctions about the middle of the well. We have then that the initial wave packet will reform itself at a half revival time later in a location mirrored about the center of the well [1]. This model will also present at certain times configurations known as \textit{fractional revivals}, which show a relocation of the initial wave packet into a number of smaller copies of itself.

The form of the wave function that we use in this work is obtained by replacing \( E_n = n^2 E_0 \) by expansion (Eqns. 7 and 11) and by considering \( n_0 \), defined as

\[ n_0 = \frac{p_0 L}{\pi} \]  

(15)

which yields

\[ \psi(x, t) = \sum_n \sin\left(\frac{n\pi x}{L}\right) \left[ \cos\left(\frac{n^2E_0 + 2n_0E_0(n - n_0)t + E_0(n - n_0)^2}{L}\right) - \cos\left(\frac{n_0E_0 + 2n_0E_0(n - n_0)t + E_0(n - n_0)^2}{L}\right) \right] \]  

(16)

3. SOUND SYNTHESIS

We investigated how we could implement a synthesis model using common sound synthesis techniques. As we wanted to sonify the particle’s position wave function, we were looking for some type of mechanism that allowed us to represent a specific mathematical function of space at each time as sound, and to control its evolution by changing the parameters of length \( L \), initial position \( x_0 \), center of momentum \( p_0 \), initial width of momentum distribution \( \alpha \), mass \( m \) and number of states in the sum \( N \).

Every eigenstate \( \phi(x) \) of the wave packet will be related to a sinusoidal sound tone according to

\[ \phi(x) = \sin\left(\frac{n\pi x}{L}\right) \Leftrightarrow \alpha_n(t) = \sin\left(\frac{n\pi t}{L}\right), \]

(17)

The wave number \( k_n = p_n/\hbar \) of each state corresponds to the frequency \( \omega_n \) of a single sound oscillator. Thus, the complete state of the quantum system, \( \psi(x, t) \), will be represented at time \( t \) by a certain sound \( S(t) \) according to the following relation:

\[ \psi(x, t) = \sum_n \sin(k_n x) C_n(t) \Leftrightarrow S(t) = \sum_n \sin(k_n t) C_n(t) \]

(18)

\( S(t) \) can be understood as a Fourier sum of sinusoidal tones whose frequency spectrum at instant \( t \) is equivalent to the momentum distribution of the wave packet at that time. Therefore, we can make a sound display of the state of the system if we are able to compute the sound waveform \( S(t) \).

We propose two different sound synthesis mechanisms for this purpose. These systems were implemented as externals for the Pure Data programming environment [6], using the PDJ framework [5].

\textbf{Wavetable approach:} Consists in computing the equation of \( \psi(x, t) \) for a discrete set of values of \( x \), storing it in a data array of length equal to the domain of \( x \), and using it as a \textit{sample waveform} of a certain duration by making the correspondence 18. For every discrete time \( t \) we construct a table of wave data (hence the name) that can be sent to a sound processing environment which will be able to read the values of the table as sound. This approach is implemented in the patch denoted as \textit{qtable}.

\textbf{Additive approach:} Consists in finding the coefficients of the Fourier expansion \( S(t) \) and use them to construct a sound signal by adding the sound of \( N \) independent sinusoidal oscillators with amplitudes weighted by these coefficients. Pure Data offers independent sinusoidal tone generators whose amplitude and frequency can be easily modified, making this synthesis method much more adequate for our sonification problem. This approach is contained in the patch labelled \textit{qexpansion}.

We now analyze in more depth the two proposed alternatives.

3.1. Wavetable approach implementation

The main steps for the sonification via wavetable synthesis are:

1. Construct a floating point table of a discrete length that equals the length of the infinite square well.
2. With a discrete index $i$ ranging from 0 to $L$, we assign each value $\phi(i,t)$ of Eqn.16 to the slot $i$ of the table. The shape and size of the wave table will depend upon the values of the parameters of the problem (Figure 2).

3. Normalize the filled table.

4. As shown in the qtable PureData patch (Figure 3) the waveform is managed as an audio sample and played-back at a rate depending on the length of the table and the sampling rate $SR$ of the digital audio system. The sound waveform is looped at a frequency of $L/\pi$. It is important to notice that the wave function is fully contained within the limits of the well and will always approach to zero near the walls. This property of the model will avoid abrupt amplitude changes that add unwanted artefacts to the synthesis model.

5. Given the distribution of probability of the particle inside of the well, we associate the same distribution to the spatialization of the sound, by determining what percentage of the sound will come out of each one of the two speakers in a stereo reproduction mode.

Figure 3. qtable patch. The parameters are controlled by managing the number atoms connected to the pdj qtable external object. The sound engine starts when the play toggle button is clicked. The time evolution of the wave packet is started by clicking on the TIME PASS ON/OFF toggle box.

2. Construct a floating point table of discrete length that equals the number of $N$ states to be considered.

3. Take an index $n$ ranging from 0 to $N-1$, and assign each value $C_n$ to the slot $n$.

4. Send the table as an list of numbers through an outlet of the external or patch.

5. As shown in the qexpansion Pd patch (Fig.5), we retrieve the list and unpack its values, storing them in separated atoms.

6. Control $N$ oscillators, where the $n$-th oscillator has a frequency of $n\pi/L$ multiplied by a scaling factor in order to make it audible.

7. Amplify the signal of $n$-th oscillator by the corresponding stored coefficient. The schema of the process up to this point is shown in Fig. 4.

8. Normalize the signal using the limiter object of Pd before sending it to the speakers.

4. ANALYSIS AND DISCUSSION

4.1. Performance Comparison

The wavetable implementation performs poorly when varying $L$ or $N$ too fast. The reason is that in the case of real-time wavetable synthesis, when $N$ is too big, computing the waveform becomes too expensive, especially if $L$ is big, due to the fact that is necessary to perform loops in the values of the position ($x = 0..L$) and in the terms of the sum $(n = 1..N)$ simultaneously. In addition, the program needs time to set the constructed table and change the table length through the built-in Java methods. Thus, the $NL$ calculations needed to write the table and the finalization of the setting methods may not be completed before the next instance of the time(b) method calls for a new table creation.
4.2. Physical analysis of the data

For each implementation, the evolution of the sound spectrum in time represents the evolution of the complete physical state. Also, the spectrograms allows the visualization of the time scales of bounded systems, such as the classical periods, the full revival times, the mirror-revival times and the fractional revival times.

**Time scales.**

The classical period of the quantum system can be observed in Fig. 6. Starting from the center of the well, the wave packet travels to the right, hitting each one of the walls and returning to the initial position $x_0$. The hits are not limited to an instant; they take up a finite amount of time to complete a rebound. The evolution of each composing frequency in the sound will depict phase bands whose slope depends in the direction of the particle’s movement. If we look at the wider simulation of Fig. 7, we can observe a symmetry over time in the state of the wave packet. Towards the end of the simulation, the wave form returns to its initial shape, so we identify this time as the revival time $T_{rev}$. The sound also exhibits a symmetry with respect to the time $t = T_{rev}/2$, being this the mirror revival time, and fractional revivals at $t = T_{rev}/3$, $t = T_{rev}/4$, and so on, but these are not displayed due to spacing limits.

**Parameter variation**

The parameter $\alpha$ represents the initial width of the Gaussian distribution of the wave packet in position space. We expect that as the width of the $x$-space Gaussian wave

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**Figure 4.** Scheme of the *qexpansion* patch. The Fourier coefficient table created by the Java external is passed as a list to the Pd patch. The coefficients are separated and used to multiply the amplitude of each sinusoidal oscillator object.

**Figure 5.** The *qexpansion* patch. The parameter and sound controls and the time evolution process are the same than in the *qtable* patch. The absolute values of the momentum distribution and the produced sound signal are stored in two different arrays.

**Figure 6.** Audio analysis of the classical period. The period completes when the packet bounces in each one of the walls and returns to the starting point. The evolution of each composing frequency in the sound will depict phase bands whose slope depends in the direction of the particle’s movement.
packet increases, and the particle becoming de-localized, the momentum distribution gets narrower around the value of $p_0$. This situation is represented by the spectrogram of Fig 8. The initial frequency bandwidth (the darker area) gets smaller with increasing $\alpha$, and decays in a narrow band around the frequency associated with $p_0$.

Figure 8. Spectrogram for a linear increase in $\alpha$. The frequency band gets narrower around the value associated to $p_0$.

The effect of varying the mass parameter $m$ is that of a variation in the group velocity of the wave packet given by $p_0/m$. It is expected that as the mass increases, the velocity decreases. This behavior can be seen to some degree in the sound spectra shown in Fig. 9, where the rate of change in the frequencies diminishes as $m$ increases. The change of the center of the momentum distribution of the wave packet given by $p_0$ is depicted in Fig. 10. It can be observed how the narrow band of frequencies around the value corresponding to the starting $p_0$ moves as a whole when it varies.

Figure 9. Spectrogram for a linear increase in the mass. The phase bands get increasingly spaced as the wave packet’s group velocity diminishes.

Figure 10. Spectrogram for a linear increase in the initial momentum. The frequency band around $p_0$ moves along with it.

The expected effect on the wave packet when changing the length of the well $L$ is that of a variation in the momenta spacing. As the parameter $L$ increases, we expect the distribution of momentum to get tighter and the eigenfunctions to lower their frequency. In Fig. 11 these situations are present, as we notice how the frequencies become closer to each other and smaller. Variation in the parameter $N$ do not affect the physical behavior of the system and only becomes relevant in a perceptual analysis of the sound.

4.3. Perceptual properties of the produced sounds

We now provide an informal perceptual-based analysis of the sonification process in terms of different perceptual attributes such as loudness, timbre and pitch of the produced sounds.

1. **Loudness** We note by looking at the wave function Eqn.16 and expressions 2 and 3 that the amplitude of the wave packet depends primarily upon the parameters $\alpha$ and at a lesser extent on $p_0$.

2. **Timbre**

   **Spectral Broadening:** We find changes in the width of the sound spectra that contribute to the sensation of body of what is listened. As viewed
in Fig. 12, the parameter \( N \) acts as the cutoff frequency selector of a kind of high-pass filter, and the combination of parameter \( \alpha \) with \( p_0 \) creates a band-pass filter effect with selectable center frequency and bandwidth.

Figure 12. Spectra for linear increase of \( N \) from 0 to 60. The cutoff frequency depends directly on \( N \).

Spectral Ordering: The phase change of the frequencies that is a product of the time evolution of the model represent the alignment in time of the sinusoidal tones composing the total sound. This is an important sound effect that is similar to a sawtooth signal in term of its perceptual quality.

On the other hand, the effect of spectral ordering of the parameter \( m \) can be noticed in Fig. 13, as the wave packet is initially localized and the resulting sound is composed by frequencies that seem very ordered. As the parameter \( m \) tends to zero, the independent frequencies disorder chaotically.

Figure 13. Spectra for change in the mass from 1 to 0 and back to 1. The ordering of the frequencies gets affected when varying the mass around small values.

3. Pitch The parameter variation that provokes the most noticeable effect on pitch is the length of the well \( L \). The state of minimal energy is defined by \( L \) and has a frequency proportional to \( 1/L \). The rest of the frequencies in the resulting sound will be integer multiples of this fundamental frequency, acting as overtones. The perceived frequency of the entire sound will then be proportional to \( 1/L \).

We also note that the parameter \( p_0 \) defines the pitch as it determines the center position of the frequency band. If \( \alpha \) is sufficiently big, the only frequency present will be the ones associated with \( p_0 \) and the perceived pitch will be the frequency of the eigenstate defined by \( n_0 = p_0\pi/L \), thus the system has the capability of a tone generator.

The system presents a large gamma of sounds that can be achieved by experimenting with combinations of parameter values. The model may react unexpectedly and present sound situations that one would not normally have guessed.

5. CONCLUSIONS

We have built a computational system that can bring quantum mechanical situations to the aural space and give the user the possibility to interact with it in several ways. In several experiments with the system, we encountered situations (mainly with slow simulation speeds) that produced attractive sounds from a melodic and rhythmic perspective. For instance, a composer could associate simple mathematical functions for the control of the parameters, and use the system for algorithmic or spectral composition: a pack of equations could be used as a controlling set and its product could be reproduced indefinitely as a musical score.

In the field of education, the model can be used in the classroom as a support to visualization methods. This perspective can extend to any physical model that presents an oscillatory behavior in time that can serve for sonification. The synthesis model can attract more minds to quantum mechanics, due to the interesting sounds that come from it, reaching students who are not otherwise being reached, encouraging self-directed learning and enabling young people to have direct involvement with purely theoretical issues, like quantum mechanics, that seem very far away from reality in the beginning.

One of the principal values of this work is that we actually constructed a digital application for the manage and
study of the sonification spectra for any physical model that presents oscillatory behavior and specially bounded state system with quantum revivals. We can graphically and acoustically explore the quantum behavior of the used model and provide a new understanding to the subject. If we want to expand this model to a problem of electromagnetic waves for example, we only would need to change the equation in the calculation method makeable() and expand or reduce the needed number of parameters, as the rest remains the same: sending the constructed table to an array in Pd and sample it. This has enormous potential, since the system acts as a general synthesis method that is already tested for a quantum mechanical model of this complexity level.

In the field of science, the sounds coming out from the system serve as tools for discovering processes that may be lost when taking a visual approach to the problem. Works like that of Saranti, Eckel and Pirr [24] show possible applications of quantum system sonification (QSS) on the field of quantum computation, by representing sound information quantum mechanically and storing it on the model of a quantum harmonic oscillator. That same idea could be worked out in our model for the bouncing wave packet by, instead of using mathematically-defined Gaussian packets, using sound information as Gaussian-shaped granules in a granular synthesis model and storing them into the potential well. Further processes that occur inside of it, and are controlled by us, would shape the information granules into new forms that could be retrieved later.

Both of the implemented sound synthesis systems confirmed to work correctly after checking that the classical period $T_{cl}$, the quantum revival time $T_{rev}$ and the quantum properties of the Gaussian wave packet modeled satisfied the theory. The system produced spectra that was according to the momentum distribution of the model, meaning that the sonification of the quantum spectra of momentum was achieved. The additive synthesis type of sonification made on the qexpansion patch proved to perform far better than the wavetable synthesis approach of the qtable patch in the aspects of computation, sound quality and sound flexibility.

In summary, we propose a sonification system that has an interesting potential of usage as a sound synthesis engine for musicians and sound related artists in general, and can help also as an interactive reinforcement for pedagogical purposes. The sonification proved to have the functionalities of a combined high-pass and band-pass filter, a stereo spatializator, and a pitched tone generator.

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7. REFERENCES