Computer music synthesis programs invite the exploration of novel rhythmic structures. Composers in this medium need not provide the referential metric pulses required by performers; nor, for that matter, contend with performers' abilities, anxieties, and hostilities when asked to produce and coordinate complex rhythms accurately. The precision with which the computer, on the other hand, executes conventional metric rhythms constitutes a second reason for exploring other approaches: the resulting performance is criticized as mechanical and "unmusical." A third factor is the opportunity for revision of musical passages after each realization. The composer may gamble in creating rhythmic textures.

Rhythmic data can be specified to the computer in a coded format that reflects the structure to be formed. Subroutines subsequently translate these data into actual starting times and durations. Such an approach enables the composer to program nonmetric rhythms in clear functional terms and to control several local events with a small number of global variables. The structure imposed by the subroutine may be simple — for example, even subdivision of a time span — or quite complex. Whether the resulting rhythm will be perceived as highly organized or, rather, as an elegantly derived random pattern depends on the particular mathematical structure involved.
Attack patterns derived from one structure, the geometric series, correspond to a musical primitive, acceleration (or its converse, deceleration). This tempo variation clearly can be heard. Further, the rate of acceleration or deceleration throughout an unfolding geometric series is constant. This may not be discerned as readily, but it does give rise to a number of systematic properties upon which an entire rhythmic language can be constructed. Precise computer control of rates of change permits acceleration and deceleration to be defined as rhythmic norms rather than as expressive deviations.

Example 1 presents the general form of a geometric series and two specific examples. Each is generated by multiplying an initial term A by a constant ratio to obtain a second term A*R, multiplying again by the ratio to produce the third term A*R**2, and continuing until N terms are determined. A minimum of three terms is necessary to define a geometric series.

EXAMPLE 1. GEOMETRIC SERIES
2. 12 18 27 40.5 60.75 (A = 12; R = 1.5; N = 5)
3. 1.25 1 .8 .64 .512 .4096 (A = 1.25; R = .8; N = 6)

Example 2 lists formulas for the sum of a series given the first term, the ratio, and the number of terms. When the ratio is between 0 and 1, this sum approaches the limit indicated in Example 3. Conversely, given the desired sum, ratio, and number of terms of a series, the first term can be determined by the formulas in Example 4. This procedure
may be used to divide a specified time span into series of
accelerating or decelerating beats. Several such series
may simultaneously divide a time span.

EXAMPLE 2. SUM OF A GEOMETRIC SERIES

\[ S = A \times (R^{*N} - 1) / (R - 1) \]
\[ S = A \times (1 - R^{*N}) / (1 - R) \]

A = first term; R = ratio; N = number of terms; S = sum

EXAMPLE 3. LIMIT OF SUM OF A GEOMETRIC SERIES

\[ \text{Lim} = A / (1 - R) \] (for ratios between 0 and 1)

EXAMPLE 4. FIRST TERM OF A GEOMETRIC SERIES

\[ A = S \times (R - 1) / (R^{*N} - 1) \]
\[ A = S \times (1 - R) / (1 - R^{*N}) \]

Interpretation of Geometric Series as Attack Points

The derivation of an attack pattern from a geometric series
is illustrated in Example 5. Six attack points are determined.
For every attack point N the attack time equals the sum of the
first N-1 terms of the series. For example, the third attack
time equals the sum of the first two terms \((1 + 1.25 = 2.25)\).
Each term of the series is represented in the attack pattern
as a time interval between consecutive attack points. Durations
may be shortened or overlapped without altering this time
interval pattern.

EXAMPLE 5. INTERPRETATION OF SERIES AS ATTACK POINTS

Series: 1.25 1 .8 .64 .512

Attack number: 1 2 3 4 5 6

Attack time: 0 1.25 2.25 3.05 3.69 4.202

\((+1.25) (+1) (+.8) (+.64) (+.512)\)
The series in Example 5 generates a rhythm that accelerates regularly from time 0 to 4.202. In a musical passage the final attack point, 4.202, generally will function as the zero point of an additional concatenated geometric series and is termed a boundary point. A time interval of .4096 is anticipated at this point as a continuation of the original series. Should the time interval that initiates the second series differ greatly from this value, the initial series will be abruptly interrupted, placing strong metrical emphasis upon the boundary point. On the other hand, should the new series begin with the anticipated time interval, .4096, whether it subsequently decelerates, or accelerates at a new rate, an elision of the two series will occur.

Reinforcement of boundary points permits the metric differentiation of attack points into strong points of arrival (boundaries) and weak passing points. The stressed boundary points may in turn project higher-level rhythmic progressions as in Example 6.* Suppression of boundary points promotes the construction of longer more fluent rhythmic gestures (Example 7).

*Examples 6-8, 10A-11B, and 14-19 are found on pages 24 through 30.
Rhythmic Relationships between Geometric Series

Three special types of rhythmic relationships exist between geometric series, each associated with a particular ratio relationship. In the first case, series share a common ratio and consequently may differ only in tempo and the number of attack points. The term transposition is used to denote the augmentation-diminution relationships between segments of such series. A second type of relationship involves series whose ratios are integer powers of the ratio of a referential base series; that is, series whose ratios are exponentially related. The third case involves retrograde-related series, the ratios of which are inversions of one another. Each of these relationships will be discussed in turn.

Transpositionally Related Series

Rhythmic transposition of a geometric series is an extension of augmentation and diminution of a rhythmic motive. A four-note deceleration at ratio 1.5 might be one such motive. It would relate not only to all other appearances of this structure at any tempo, but to all four-note subsegments of decelerations at ratio 1.5 with greater numbers of attack points.

Within a series transpositional relationships exist between segments with equal numbers of attack points, whether they overlap one another, follow each other immediately, or are separated by other points of the series. In the eight-term series shown in Example 8, the seven-term segment, terms
2 through 8, is a transposition by R of terms 1 through 7.
The two canonic voices proceed in rhythmic unison, but the second is an augmentation of the first. Example 9 illustrates several transpositional relationships between segments of a lengthy accelerating series. Each rhythmic transposition is associated with pitch transposition or inversion.

EXAMPLE 9. TRANSPOSITIONALLY RELATED SEGMENTS

Basic series of pitch classes (P.C.) accelerates at ratio R.

<table>
<thead>
<tr>
<th>P.C.</th>
<th>0 1 5 6 10 11 3 4 8 9 1 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time Intervals</td>
<td>1 R  R² R³ R⁴ R⁵ R⁶ R⁷ R³ R⁹ R¹⁰ R¹¹</td>
</tr>
</tbody>
</table>

Six-term segments: Transposition by R⁶ and interval 3

| 0 1 5 6 10 11 | ...³... 3 4 8 9 1 2 |
| 1 R  R² R³ R⁴ R⁵ *R⁶ ... R⁶ R⁷ R³ R⁹ R¹⁰ R¹¹ |

Four-term segments: Transposition by R⁴ and interval 10

| 0 1 5 6 ...¹⁰... 10 11 3 4 ...¹⁰... 8 9 1 2 |
| 1 R  R² R³ *R⁴ ... R⁴ R⁵ R⁶ R⁷ *R⁴ ... R⁸ R³ R¹⁰ R¹¹ |

Rhythmic transposition by R² (overlapping): P.C. transposition by 5

| 0 1 5 6 |
| 5 6 10 11 |
| 10 11 3 4 8 9 |

Rhythmic transposition by R; RI-related pitch-class groups

| 0 1 5 6 10 |
| 1 5 6 10 11 |
The six sets of attack points in Example 10A result from six transpositionally related series with ratio 2. Three systematic properties are of interest. Short dotted lines indicate the canonic relationships between lines 1 and 2 and between lines 3 and 4. These occur whenever the first time interval of one series equals R times the initial time interval of another. Also of note is the common attack point 35 between lines 5 and 6. This is a systematic result of their respective common attack points with line 1 at times 7 and 15. Finally, note the rearrangement of lines 1 through 4 in Example 10B. The rhythmic motive formed by the second attack points of the four lines recurs in augmentation beginning at times 3 and 7. This is a general result of the transpositional relationship between the four lines; it is not dependent upon their being geometric series.

Exponentially Related Ratios

In Example 11A terms in the series 1 R R² ... are grouped in pairs to obtain the time intervals 1*(1+R), R²*(1+R), R⁴*(1+R), and R⁶*(1+R). Thus a series with the ratio R² is embedded within the original series. If terms of the basic series are grouped in threes and fours, series with the ratios R³ and R⁴ are formed. All these series utilize attack points of the original series and thus may be projected from it by registral, dynamic, or other means. Example 11B is one such passage. The upper voice of this geometric Alberti bass articulates an R² deceleration; the middle and lower voices state
R^4 progressions. Subsequent appearances of series with ratios R^2 and R^4 will relate transpositionally to these embedded series.

Retrograde-Related Series

Retrogression of a series with ratio R produces a related series with the inverse ratio 1/R as in Example 12. When two so-related series are interpreted as attack-time series beginning simultaneously, a palindrome results. (This symmetrical structure is always formed by the combination of a rhythm with its retrograde.) Example 13 illustrates the formation of a palindrome from the series in Example 12. The resulting aggregate attack rhythm generally consists of a central unpaired time interval surrounded by symmetrical pairs as in Example 13.

EXAMPLE 12. RETROGRADE-RELATED SERIES; INVERSE RATIOS

A. .16 .24 .36 .54 .81 \begin{align} R &= 3/2 \end{align}
B. .81 .54 .36 .24 .16 \begin{align} R &= 2/3 \end{align}

EXAMPLE 13. PALINDROME FORMATION

Series are those given in Example 12

A. 0 .16 .40 .76 1.30 2.11
B. 0 \begin{align} \begin{bmatrix} .81 & 1.35 & 1.71 & 1.95 & 2.11 \end{bmatrix} \end{align}

Aggregate time-interval pattern:

\begin{align} \begin{bmatrix} .16 & .24 & .36 & .05 & .49 & .05 & .36 & .24 & .16 \end{bmatrix} \end{align}

The recurrence of time intervals may be used to reinforce pitch associations. In Example 14 the computer instruments T vibes and Rca articulate a pitch canon. Rhythmically, in each measure the Rca line retrogrades the T vibes line. Arrows
indicate pairs of equal time intervals that present related pitch successions. Similar pairs are present in each measure, but the proportions between them vary. In Example 15 similar pitch dyads articulated by equal time intervals are circled for each measure. These time intervals vary as the measures themselves are successively longer and the ratios within them progressively more extreme. The simultaneous attacks that articulate the beginning of each new measure form a decelerating geometric series.

**Restricted Sets of Ratios**

A single set of ratios, integer powers of the twelfth root of 2, gives rise to all the relationships among geometric series discussed previously. Deceleration ratios are powers of the base ratio \(2^{**(1/12)}\); acceleration ratios are powers of the inverse \(2^{**(−1/12)}\); and the equal-note-value ratio 1 is obtained through exponentiation by 0. Series with more highly exponentiated ratios are embedded in series with lesser-exponentiated ratios; for example, series with the ratio \(2^{**(6/12)}\) are embedded in series with ratios \(2^{**(1/12)}\), \(2^{**(2/12)}\), and \(2^{**(3/12)}\). Each ratio \(2^{**(n/12)}\) can be paired with its inverse \(2^{**(−N/12)}\) to obtain retrograde-related series. Transposition occurs with the repeated use of ratios and even in the partition of a single series. Finally, sequential relationships result when the durations of successive measures form progressions defined by the same ratios.
Independent Tempo Control

In the MUSIC4 and MUSIC360 programs, accelerations are normally programmed with a tempo card. Tempi are specified for the initial and terminal points of a passage. The programs determine the durations between these points by proportional interpolation. The sum of these durations, that is, the total duration of an accelerating passage, cannot be directly controlled. Using geometric series, one can specify this overall time span and additionally control the rate of acceleration. This also permits several voices within a complex rhythmic texture to proceed in strict tempo while others are varied. The programming is able to process subdivisions of beats, tied notes, and rests. In Example 16 equal subdivision of beats is maintained in an accelerating context. Terraced rhythms are formed.

Sequential Rhythms

Geometric rhythms may be used to program more complex sequential progressions. In Example 17 each measure is subdivided by a four-term series with ratio $R$. The overall durations of the measures are chosen to form a series with ratio $R^2$. This results in the repetition of the second pair of time intervals in each measure by the first pair in the following measure. Two rhythmic structures are sequentially transposed: the measure, and the cross-measure repeating time interval pairs.

In Example 18 measure durations form successive 5 to 4 relationships. Nine-term equal-note-value series result from
metric modulation in which the new eighth note equals the previous quintuple eighth. A four-measure twelve-term acceleration completes the example.

Extended Geometric Series

In the preceding examples geometric series subdivide short concatenated time spans. The series are also effective when extended across long time spans. The combination of several accelerating and decelerating series, each with several attack points and all beginning and ending together, creates an archlike aggregate rhythm with rapid directed motion at the opening and close, and relatively little activity in the middle.

Geometric-Metric Combination

Example 19 illustrates two geometric rhythmic structures that map onto the regular pulses of metrical music. Both utilize the 2 to 1 relationships formed between every second time interval generated by series with the ratios $2^{**}(0.5)$ or $2^{**}(0.5)/2$. In the first passage two retrograde-related four-element series and the following boundary point together articulate four consecutive beats. In the second passage six of eight beats are stated. Other fractional powers of simple integers, for example $2^{**}(1/3)$ or $3^{**}(1/2)$, may be used as ratios to articulate metric beats. Such patterns might provide cues to performers in compositions for tape and instruments.
Another approach to the mapping of geometric structures onto metric beats is a rapid computer ratio estimation program. A ratio may be determined, for example, which generates a seven-term series whose first four time intervals total 1 second and remaining intervals total 2 seconds. If a final boundary point is included, this series places attacks on three of four successive metric beats.

To conclude, I do not propose geometric series as a rhythmic language for everyone, nor myself as Moses leading his people into the rhythmic promised land. Rather, I present these series as an example of an approach to the computer in music composition in which careful investigation and structuring of a musical domain leads to fruitful compositional results. I adhere to a two-part assumption contrary to much that has been presented at this conference: that the proper role of the computer is to perform our music, with accuracy and elegance; and that our proper and exclusive role is to compose.

Note: Geometric series and their compositional applications are discussed at length in my doctoral thesis, Rhythmic Applications of Geometric Series. This is available from University Microfilms, Ann Arbor, Michigan, and includes scores and discussions of the compositions Exercycles and Points in Time, both of which employ geometric series exclusively, and FORTRAN programming for the implementation of these series.
EXAMPLE 6. STRESSED BOUNDARY POINTS

Ratios of decelerations are indicated in brackets. Measures increase in duration at ratio 1.122.

EXAMPLE 7. ELIDED GEOMETRIC SERIES

The initial time intervals of measures 2 through 6 continue the progressions of the previous measures. This determines the relative durations of the measures.
EXAMPLE 8. RHYTHMIC TRANPOSITION

Voices decelerate at ratio $R$ in rhythmic unison.
Lower voice is an augmentation of upper voice by $R$. 
EXAMPLE 10A. INTERACTIONS OF TRANSPOSITIONALLY RELATED SERIES

1) 0 .1 .3 7 15 .31
2) 0 2 6 14 30
3) 0 1 4 10 22.5
4) 0 3 9 40 35
5) 0 7 15 35
6) 0 5

EXAMPLE 10B.

1) 0 1 3 7 15
2) 0 2 6 14 21
3) 0 1 4 10 22.5
4) 0

(R=2)

(1) 1+R

(3) 1+R

(7) 1+R+R^2
EXAMPLE 11A. EMBEDDED SERIES: EXPONENTIALLY RELATED RATIOS

Basic series: 1 R R² R³ R⁴ R⁵ R⁶ R⁷

Embedded series with ratio R²:

A. \( (1+R) (R²+R³) (R⁴+R⁵) (R⁶+R⁷) = \)
   \( 1*(1+R) R²*(1+R) R⁴*(1+R) R⁶*(1+R) \)

B. \( (R+R²) (R³+R⁴) (R⁵+R⁶) = \)
   \( 1*(R+R²) R²*(R+R²) R⁴*(R+R²) \)

EXAMPLE 11B

Basic series decelerates at R. Upper line decelerates at R².
Middle and lower lines decelerate at R⁴.
Example 14

*Ratios are integer powers of \(2^{1/2}\); e.g. \([+6]\) represents \(\sqrt{2}\).

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Example 15

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EXAMPLE 16. EQUAL SUBDIVISION OF ACCELERATING BEATS

Ratio = .8901 (2**(-1/6))
EXAMPLE 17. SEQUENTIAL PROGRESSIONS

Durations of measures form \( R^2 \) progression.
Durations within measures form \( R \) progressions.

EXAMPLE 18. SEQUENTIAL METRIC MODULATION

Durations of measures form progression with ratio \(3 \) (\(4:5\)).
EXAMPLE 19. MAPPING GEOMETRIC SERIES ONTO METRIC PATTERNS

\[
\begin{align*}
R &= \sqrt{2}/2 \\
1 & \quad 3 & 1 & \quad R &= \sqrt{2} \\
2 & \quad 1 & \quad 4 & \quad R &= \sqrt{2}/2 \\
3 & \quad 2 & \quad 1 & \quad 5 & \quad b7 \\
4 & \quad 1 & \quad 1 & \quad 1 & \quad 1 & \quad R &= \sqrt{2}/2
\end{align*}
\]