The non-linear distortion (waveshaping) is now a well-known technique which has been previously described (1,2). A sine wave is distorted (waveshaped) by using a non linear transfer function $y = f(x)$. This produces harmonic distortion, the amplitudes of which is dependent upon both the nature of the distortion and the amplitude of the input.

1. If the non linear transfer function $y = f(x)$ is a polynomial of degree $n$, no more than $n$ harmonics can be produced (however some may be insignificant).

2. It is possible to choose, and compute, a polynomial which gives a specific harmonic distribution at the output, when the input amplitude is given.

3. There is a complete independance between odd and even harmonics, which are respectively produced by odd and even terms of the polynomial.

4. The harmonic distortion is ordinarily increasing when the input amplitude is growing. So that this input can be called an index of time but it also governs the output energy.
A TRANSFER FUNCTION

\[ y = d_0 + d_1 x + d_2 x^2 + \cdots + d_n x^n \]

MORE A MORE AS DISTUHO
example 1:

A first example, taken from "Voyelles d'Evell" is a clarinet-like sound. The polynomial, of 85th degree, simulates a saturation and has been computed by giving a point where all first derivatives are zero. This produces sounds with only odd harmonics, and no foldover if the sampling rate is well chosen.

These "notes" are played following an harmonic scale. In the second part of this melody, the first note of the triplet is delayed and the third one is advanced, so that at the end, we hear a real chord on harmonics 1, 3, 5 of a new note.
Example 2:
A possible development of waveshaping is to combine it with amplitude modulation. It is realized by multiplying the output of a normal waveshaping instrument by a "carrier wave", which shifts the harmonic distortion around each of the components of the carrier. It is very close to the FM technique, with the additional advantage of a new control upon the partials distribution (via the choice of a transfer function). An harmonic spectrum is produced when the carrier/modulation frequency ratio is a rational. An harmonic percussion (touelles d’eveil) last sound) has been obtained with a complex carrier which is modulated by a distorted wave.
The classical use of non-linear distortion can lead to many sounds, and the amplitude modulation is helpful to provide inharmonic sounds. However, a special class of polynomials is what is called Chebyshev polynomials. A Chebyshev polynomial of order $n$ is simply a transfer function which looks like a warped sine wave. When the input is of amplitude 1 and frequency $f$, the output is of amplitude 1 and frequency $n f$. However, for different values of the input amplitude, the output contains harmonics ranging from 1 to $n$. 

The diagram shows a block diagram of a circuit labeled "DIST" with various inputs and outputs, and a Chebyshev function $x^5$ plotted on a graph.
example:
When a sine wave is distorted by a Chebyshev polynomial following
the envelope of our figure, the output goes from frequency f to f.m.
The amplitude of this output is now decreased in its second part,
and serves as an input for a new waveshaper. The phenomenon is a bit
complex, but at the middle point we obtain f.m. and at the extremities
f.m and f.m (at least for the ear).
In "Le Souffle du Doux" two sequences are derived from this scheme, with adequate envelopes to "catch" the middle point. The composition of the first sequence (beginning of the piece) is obtained with increasing values of m,n, which are in fact also reflected by high-pitched random noise some octaves higher. By this way we obtain a crescendo, despite the fact that the initial frequency is always the same (an E drone). The second sequence is more like an algebraic game, at least at its beginning (3,5,7,9...). At this particular point (7,9,63) the evolution is particularly rich, despite the fact that the instrument is always so simple.
example 5:
We can use transfer functions which are different from polynomials. We run the risk of having "infinite distortion", that is to say foldover. When the input frequency is very low, this does not happen easily. For example we can use trigonometric functions, which produce Bessel terms. In fact it is FM, with a carrier frequency of zero value. The use of three notes with very close frequencies produces beats at an irrational rate, giving some life to these otherwise static sounds. This allows the sound to be used like a drone during three minutes, in Le Souffle du Dous.
example 5:
Above this drone, some exploding whistle come from a random frequency modulation of distorted sine wave. If the transfer function is a Chebyshev polynomial, the predominant harmonic partials seem to succeed each other regularly, giving a timbre melody. In le Souffle du Ross, successive polynomials are used which are derived from the 11, 10, 9 and 15, 14, 13, 12 series by taking their "octave root" (divide them by two as long as possible). The timing of the sequence is simple: the time between events is exactly the order of the polynomial. The duration of the events is half the order in the first sequence, and the order itself in the second one.

\[
\begin{align*}
T_c &= 62.5 \times H_b \\
T_f &= -1.0 \times T_c \quad (-50\%) \\
T_e &= T_e
\end{align*}
\]
example 6:

'L'Approche de la nuitre is a slow piece where there are very few events, but rather a sort of internal evolution if timbre. The first and last movements use the same instrument: the amplitude of the waveshaping instrument (right part) is controlled by another distorted signal. This signal is very slowly evolving, because there is only one cycle for the oscillation of the sine wave during four minutes, and its variation is almost, but not quite, regular. By using different Chebyshev polynomials these amplitude controls correspond to an automatic mixing in this sense that they are growing and decreasing in an algorithmic way.'
A particular use is the "square wave". If the input frequency is \( n \) and if we use a Chebyshev polynomial of same order, we have harmonics ranging from \( 0 \) to \( n^2 \). The use of six different notes provides enough musical complexity.

\[
\begin{align*}
P_2 &= H_3 & P_4 &= H_5 & P_6 &= H_7 & P_8 &= H_9 & P_{10} &= H_{11} \\
\text{Cheby} 3 & \quad \text{Cheby} 5 & \quad \text{Cheby} 7 & \quad \text{Cheby} 9 & \quad \text{Cheby} 11 \\
H_3 & \quad H_5 & \quad H_7 & \quad H_9 & \quad H_{11}
\end{align*}
\]
The first sequence simulates a choir effect by using a slow and narrow random frequency modulation upon each note, a technique suggested to me by John Chowning.

\[ \varphi_0 = 41.3 \text{ Hz} \]
\[ \text{random} \quad \% \quad \varphi_1 \rightarrow \varphi_2 \quad \text{sampling rate} \quad 55 \text{ kHz} \]
\[ \text{random} \quad \% \quad \varphi_3 \rightarrow \varphi_4 \]
\[ \text{random} \quad \% \quad \varphi_5 \rightarrow \varphi_6 \]

"Choir" Effect

The last sequence is a providential gift of our best enemy: the foldover. If the fundamental frequency is high enough and the sampling rate low enough, so many components come together in the audio domain that you don't know who is who.

\[ \varphi_0 = 388.6 \text{ kHz} \]
\[ \text{random} \quad \% \quad \text{sampling rate} \quad 48 \text{ kHz} \]

"Water Fall" Effect
Conclusion:
I have attempted to show how a technique such as non-linear distortion can be used in a musical way. It seems to me that two things are necessary: I/ an additional control upon the microstructure of the sound, which determines one or another aspect of timbre 2/ a compositional idea, which gives a rhythm, an harmony of timbre.

All the examples here described come from Violelles d'Étoilé, le Souffle du Goût, and l'Approche de la lumière, which are available on record (3).

References:
The basic technique, and a bibliography, is found in
(1) D. ARFIB: Digital synthesis of complex spectra by means of multiplication of non linear distorted waves. JAES, 1979 October, volume 27, number 10, pp 757-760.
(2) M. LEBRUN : Waveshaping synthesis. JAES, 1979 April, volume 27, pp 250-256.

(3) D. ARFIB: Musique numérique, available at La Comète Sensible, 5 allée Albeniz, 13000 Marseille France.

APPENDIX A: COMPUTATION OF CHEBYSHEV POLYNOMIALS (FORTRAN)

DIMENSION A(400), TCHE(512)

20 A(I) = 0.540
40 X = PI/I*xi(I)/RAS.
20 A(1) = A
40 A(2) = X
60 CONTINUE 3 A(I) = X*A(I-1) + A(I-2)
80 N = THE ORDER
100 TCHE(I+1) = A(I+1)
120 TCHE(I+2) = TCHE(I+1)
140 END

THE IS A TABLE THAT CONTAIN THE TRANSFER FUNCTION

511