A Modified Rectangular Waveguide Mesh Structure with Interpolated Input and Output Points

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Abstract
The rectangular waveguide mesh presents aspects of redundancy when computing the impulse response of an ideal resonator. Its structure is thus modified, to define a new structure where redundant computations and unnecessary memory consumption are removed. The modified mesh saves half of the memory and computations, meanwhile it preserves all the numerical properties of the rectangular waveguide mesh accounting for stability, direction-dependent propagation speed of the wavefronts and so on. A general method, which is derived by bilinear interpolation and deinterpolation, is adapted for generalizing the input/output point positions in the modified structure. Simulations confirm the conjectures advanced herein.

1 Introduction
Waveguide meshes (Van Duyne and Smith 1993), in their several formulations, are deserving a certain attention by researchers concerned with the design of resonators for musical, audio and multimedia applications. Previous studies (Fontana and Rocchesso 1995; Savioja and Välimäki 2000), for instance, have inspired some interesting applications of the waveguide mesh (from now denoted as WM) for modeling percussion instruments (Fontana and Rocchesso 1998; Aird et al. 2000), string instrument bodies (Huang et al. 2000), and in the analysis of reverberant enclosures (Savioja et al. 1995; Murphy and Howard 2000).

Although some formulations of the WM allow to realize models that closely approach the behavior of an ideal resonator, the original structure (Van Duyne and Smith 1993) remains a valuable trade-off between computational efficiency and versatility. Moreover, it allows a straightforward implementation, and this is very useful at least when one conducts preliminary tests of reliability of a WM model with respect to a given modeling problem.

In that case, minimizing the computational cost of the procedure is often an important issue. Any redundancy introduced in the computations by a heavier implementation of the model translates into unnecessary hardware usage, and longer wait for the output. This is true especially when the WM is used in problems involving accurate modeling of huge resonators: in this case, processing time and memory consumption often become a critical factor.

The rectangular (square) WM (SWM from now) exhibits this redundancy. This can be noted, for example, by analyzing the spectrum of its magnitude response: such a response in fact mirrors at half of the Nyquist frequency, independently from where the impulse is injected and the response is taken, suggesting that half of the computations performed during the calculation of this response are unnecessary. We show that, in these cases, an SWM can be turned into a similar, lighter structure saving half of the memory and computations. In spite of this, the performance of the SWM is preserved, since only redundant information is removed by the modified structure.

Moreover, a method to calculate the output in correspondence of a point that is misaligned from the junction positions is proposed, together with a strategy for injecting a sig-
nal when the input position differs from any scattering point position. Such a generalized excitation and acquisition points can be obtained by applying bilinear interpolation to the existing scattering points in the modified SWM (Savioja and Välimäki 1997). Though, the interpolation and deinterpolation formulas must be rewritten for accommodating them to the modified structure.

An alternative treatment of these arguments, based on the theory of Digital Waveguide Networks, will be cited whenever it has contributed to inspire this research, and when the conclusions are similar (Bilbao 2000). Rather, the background of the present work is more focused on the theory of audio signals.

2 A Modified SWM

The impulse response of an SWM is known to present redundant information, in a way that its spectrum mirrors at half of the Nyquist frequency. This characteristic does not depend on the position of the junction where the impulse is fed, and the position from where the response is taken. From a signal-theoretic viewpoint, this descends from the fact that the discrete-time transfer function which describes the response of the SWM with respect to the input and output positions is always a function of the variable $z^{-2}$ (Van Duyne and Smith 1993). Since this property comes from the numerical scheme which is computed by the mesh, it appears also in a realization made adopting a Finite Difference Scheme.

This issue has been outlined also by Bilbao (Bilbao 2000), who shows that the SWM can be subdivided in two mutually exclusive schemes. Their independence comes from the fact that the SWM processes two subsets of data at each time sample, and never performs any kind of merging between them. As a result, a sample which is taken from one scattering point at a given time step does not have structural relations with the sample which is taken at the previous (or following) time step from the same point.

The excitation in general establishes a mutual dependence between adjacent samples, so that the output from the SWM is in most cases informative over the whole spectrum. In spite of this, there are situations where one of the two subsets becomes unnecessary. In particular, the impulse response comes from the excitation of only one of them, hence the spectral content coming from the samples belonging to the latter subset, which are always equal to zero, is redundant.

Consider an SWM as a connection of objects, each one made of one $N$-port scattering junction and $N$ outgoing branches containing one unit delay\(^1\). Now, modify the mesh structure by interleaving these objects with new ones, where the unit delays have been substituted by delay-free connections.

\(\text{Figure 1: Modified SWM (size } 5 \times 5\text{) having reflecting edges.} \)

\(\bullet \) are scattering junctions belonging to $S_{2n}$. $\blacksquare$ are scattering junctions belonging to $S_{2n+1}$.

Denote the set containing the former objects with $S_{2n+1}$, and let $S_{2n}$ be the set containing the new ones. Figure 1 depicts the structure resulting from an SWM of size $5 \times 5$ having perfect reflection at the boundaries.

At each time step, junctions belonging to $S_{2n}$ compute the signal in the same way as it happens in the normal SWM. Yet, wave signals are sent without delay back to the junctions belonging to $S_{2n+1}$. Hence, samples which do not keep information are overwritten with values that, in the normal SWM, are computed during the following time step (see figure 2).

\(\text{Figure 2: Interpretation of the signal generated by the modified SWM.} \)

Samples produced by the junctions in $S_{2n+1}$ follow the samples produced by the junctions in $S_{2n}$.

\(1\)Clearly, it will be $N = 4$ for the SWM, and $N = 6$ for the 3-D rectangular WM (Savioja et al. 1995).
The stability and propagation properties of the modified structure are the same as in the SWM, since the numerical scheme realized by the modified mesh includes the scheme computed by the SWM. From a signal-theoretic viewpoint, the information which is present in the modified structure equals the information conveyed by the SWM.

The modified SWM is convenient whenever the excitation signal mirrors at half of the Nyquist frequency. Under this assumption, the global amount of memory in the new structure is halved, while the number of operations needed to calculate the output is in principle the same. Though, considering that two time samples of the output are calculated during each computation cycle, the number of operations that are needed to compute the output signal is reduced by a factor of two as well. Such a convenience appears in an FDS realization of the modified scheme as well. In this case, the reduction in memory consumption appears in the form of just one unit delay associated with each scattering point, instead of two.

Figure 3: Frequency response of a SWM (size 25×25). Excitation at the center.

Simulations conducted over the two models confirm that the spectral redundancy present in the magnitude response of the SMW (figure 3) is removed by the modified structure (figure 4).

Figure 4: Frequency response of a modified SWM (size 25×25). Excitation at the center.

3 Interpolated Input and Output Points

The modified structure can be excited in correspondence of the junctions belonging to \( S_{2n+1} \). Likewise, each one of these junctions presents the impulse response calculated in correspondence of its own position. In spite of this, there are cases where the excitation or acquisition points cannot correspond with the positions of the scattering points.

In these cases, first-order Lagrange interpolation can be used to interpolate between junctions. This method can be extended to the two-dimensional case in the form of bilinear interpolation, whose versatility and ease of use together with the SWM have been previously shown (Savioja and Välimäki 1997).

In that treatment, the formulas accounting for coefficients \( w_{ij} \) that weight the samples \( s_{ij} \) of the (four) nearest scattering points, namely \( 11, 12, 21, \) and \( 22 \), are the following ones:

\[
\begin{align*}
    w_{11} &= (1 - x)(1 - y), \\
    w_{12} &= x(1 - y), \\
    w_{21} &= (1 - x)y, \\
    w_{22} &= xy
\end{align*}
\]

where \((x, y)\) are the coordinates of the interpolated output \( v \) relative to position 11:

\[
v(x, y) = \sum_{i=1}^{2} \sum_{j=1}^{2} w_{ij} s_{ij}
\]

The same weighting parameters are used to deinterpolate the input \( u(x, y) \) over the nearest scattering points:

\[
s_{ij} = w_{ij} u(x, y)
\]

All these formulas hold when the digital waveguide lengths have been normalized to unity.

Such relations apply also to the junctions belonging to \( S_{2n+1} \) in the modified structure, once they are reformulated (see figure 5) in new coordinates \((x^*, y^*)\). This transformation consists of a translation by \((-1/2, -1/2)\) followed by a rotation by \( \pi/4 \). The new coordinates are finally stretched by a factor equal to \( \sqrt{2} \). New weighting parameters are thus found out, with the notations depicted in figure 5(b):

\[
\begin{align*}
    w_N &= (1 - x^* + y^*)(1 + x^* + y^*)/4 \\
    w_W &= (1 - x^* + y^*)(1 - x^* - y^*)/4 \\
    w_E &= (1 + x^* - y^*)(1 + x^* + y^*)/4 \\
    w_S &= (1 + x^* - y^*)(1 - x^* - y^*)/4
\end{align*}
\]
Figure 5: Reformulation of bilinear interpolation from original (a) to new (b) coordinates obtained by translating, rotating and stretching the old set.

Suppose, for example, to swap $S_{2n+1}$ and $S_{2n}$ in the square structure that produces the spectrum depicted in figure 4; we obtain a new, square mesh that is centered around a junction belonging to $S_{2n}$, like the one depicted in figure 1. Clearly, in this case an excitation at the center can no longer be applied. If we excite the center of the mesh using deinterpolation over the four neighbors, a response is obtained like the one (in solid line) in figure 6, where the non-deinterpolated excitation (refer to figure 4) is repeated (in dashed line) for ease of comparison. Note that some amplitude distortion appears, coming from Lagrange interpolation. In this case, distortion is at its maximum since we are interpolating on a point which is farthest from any neighbor scattering junction. In general cases amplitude distortion is less severe.

Figure 6: Spectrum from deinterpolated excitation around the center of a modified SWM sized $25 \times 25$ (solid line) compared with spectrum depicted in figure 4 (dashed line).

Finally, note that (de/)interpolation can be applied, more in general, to any difference scheme whose scattering nodes do not match with the excitation/acquisition point positions.

4 Conclusion

A modified rectangular waveguide mesh has been presented. It processes a numerical scheme which is included by the square waveguide mesh. It saves half of the computations and memory without loosing any information during the calculation of the impulse response. In the case that the input and output points differ from the junction positions, bilinear deinterpolation and interpolation can be respectively applied to feed the mesh and acquire its response.

The analysis presented in this paper can be easily extended to 3-D rectangular waveguide meshes and to finite difference schemes.

References


