Modeling of Woodwind Bores with Finger Holes

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Abstract

Digital waveguide modeling has turned out to be a promising method for constructing efficient computational models of woodwind instruments. Typically there has been many audible deficiencies in the sound produced by these real-time synthetic models and, furthermore, the variety of playing techniques has been very limited. These problems arise in part from the simplified model for the bore. In this paper new methods for including finger holes and losses in waveguide models of woodwind instruments are introduced. Each finger hole is modeled as a three-port junction in the waveguide. The position of the holes is adjusted by means of fractional delay fine-tuning techniques, such as FIR-type interpolation and deinterpolation.

1 Introduction

Finger holes (or tone holes, or side holes) are a unique feature of woodwind instruments. They have two essential functions: (1) controlling the fundamental frequency and (2) influencing the sound quality. The effect of open finger holes to the sound is characterized by the open-hole lattice cutoff frequency [Benedix, 1960], [Benedix, 1976]. Above this frequency the sound waves mainly radiate out of the tube and produce only weak resonances in the tube. Furthermore, even closed tone holes have their effect, as they effectively lengthen and enlarge the bore in their vicinity [Benedix, 1960], [Benedix, 1976].

So far the tone holes have been neglected in digital waveguide models for woodwind instruments [Smith, 1986], [Vilimäki et al., 1992a], [Vilimäki et al., 1992b]. The obvious limitation in these models is that the fundamental frequency of a synthesized tone can only be controlled by changing the length of the delay lines must form the digital waveguide model for the bore. From the viewpoint of the sound wave propagating inside the bore, this is a very unnatural procedure. Dropping or adding unit delays causes a discontinuity to the signal which can be heard as an annoying click.

Another deficiency of simplified woodwind bore modeling is that the tone-hole lattice cutoff effect is not automatically included in the model. Thus, its effect has to be added separately, usually during the design of the radiation filter.

In addition to the above mentioned problems, the lack of tone holes limits the playing styles that are available for the user. The multophonics, for example, cannot be produced by a model without finger holes. When proper finger hole models are included the multophonics as well as other special effects, like the key claps, become available for the user of the computer model such as they are for the player of a real instrument.

In a complete model for a woodwind bore to the viscous and thermal losses have to be taken into account. These losses attenuate the resonances of the bore and thus have certain effect on the sound.

In this paper it is shown how models for the 'finger holes and losses can be included in a waveguide model for a woodwind instrument. The paper is organized as follows. In Section 2 the principles of waveguide modeling are recapitulated and the use of fractional delays in the context of digital waveguides is discussed. A model for an open tone hole and a related scattering junction are studied in Section 3. Implementation issues and the effect of approximations associated with fractional delay techniques are also discussed. An approach for approximating the effect of losses by means of a digital filter is introduced in Section 4. Section 5 concludes with remarks and directions for future research.

2 Digital Waveguide Modeling

2.1 Basics

A good tutorial on waveguide modeling has recently been written by Smith [1993]. A more theoretical description of waveguide principles and applications can be found in Smith [1987].

The digital waveguide modeling technique can be applied to the simulation of one-dimensional physical
waveguide, e.g., a narrow tube, a vibrating string, or a long and thin bar. For woodwind instruments like the flute or the clarinet the assumption of one-dimensional wave propagation is valid for frequencies below about 10 kHz.

The general wave equation describes the wave behavior by means of a partial differential equation. Its famous solution due to d'Alambert tells us that a one-dimensional wave can be divided into two components that propagate in opposite directions. Digital waveguide modeling is based on this idea.

The basic building block of a waveguide model is a digital waveguide which consists of two delay lines that store and shift samples of the two signal components. In other words, a digital waveguide is a bidirectional delay line.

2.2 Fractional Delay Filters

A limitation in the basic waveguide modeling scheme is that the length of the delay lines is impossible to adjust more accurately than by integer multiples of unit delays when the sampling rate is kept constant. The solution to this problem is the concept of fractional delay (FD), i.e., a noninteger digital delay.

An ideal fractional delay would be one that delays all the different frequency components of a signal with the same desired delay without amplifying or attenuating any of them. This means that a linear-phase allpass filter is called for. Unfortunately, such a digital filter is in general not realizable. The reason for this is that the impulse response of the ideal FD filter is infinitely long. Evidently the fractional delay has to be approximated somehow.

For both FIR and IIR fractional delay filters, a maximally flat approximation can be designed in closed form. For the kind of approximation the error and its N derivatives—where N is the order of the filter—vanish at a certain frequency, this time at 0 Hz. The maximally flat FIR filter approximation for the fractional delay is, surprisingly enough, equivalent to Lagrange interpolation which is a well-known polynomial interpolation technique in numerical analysis. The filter coefficients for the Lagrange interpolator are expressed as (Laine, 1988):

\[ h(n) = \sum_{k=0}^{N} \frac{D-k}{n-k} b_k u_k \quad (n=0,1,\ldots,N) \]  

where \( D = \text{floor}(D) + 1 \) is the desired delay with the fractional part \( D \) at \( n \), and \( N \) is the order of the filter. For the design of a maximally flat allpass filter see (Lauluk et al., 1993). A comprehensive tutorial on FD filter approximations can be found in (Lauluk et al., 1993).

In the following section, application of FIR-type interpolation techniques for digital waveguide modeling is studied.

2.3 Fractional Delay Waveguide Filters

Fractional delays are needed in many DSP application areas but in digital waveguide modeling of musical instruments they are essential. This is a consequence of the fact that waveguide models mainly deal with propagation delays that are not integer multiples of the sampling interval. When FD approximation is utilized spatially continuous discrete-time simulation is achieved.

In practice fractional delays are needed for two cases in digital waveguide modeling: (1) in cancelling the length of a digital waveguide, i.e., in interpolating the output of both delay lines, and (2) for connecting two or more waveguides together at arbitrary points. The former case has been used for several years. For example, Jaffe and Smith (1983) used a first-order allpass filter to tune a waveguide string model, and Karjalainen and Laine (1999) showed how Lagrange interpolation can be applied to the same problem.

The latter approach was introduced in (Välimäki et al., 1993) where it was shown how two digital waveguides with different impedances can be cascaded in such a way that the connection lays between the sampling points. This technique can be employed, e.g., in articulatory speech synthesis which is based on a physical model for the human vocal tract.

The two basic operations employed in fractional delay waveguide models are FIR-type interpolation and deinterpolation. They are illustrated in Fig. 1. Interpolation is nothing more than FIR filtering using interpolating coefficients, and the approximation for the ideal interpolated signal value \( u(n-D) \) is thus computed as

\[ u(n-D) = \sum_{k=0}^{N} h(k)u(n-k) = h^T u \quad (n=0,1,\ldots,N) \]  

where \( h = [h(0) \quad h(1) \quad \ldots \quad h(N)]^T \) is the coefficient vector of the FIR interpolator for the fractional delay \( D \) and \( u = [u(n) \quad u(n-1) \quad \ldots \quad u(n-N)]^T \) is a vector of signal samples to be interpolated.

Deinterpolation (Välimäki et al., 1993) is implemented with the transpose structure of an FIR

![Fig. 1](attachment:image.png)
interpolator. It can be expressed as

$$\tilde{u}_n = u_n + bu(n-D)$$  \hspace{1cm} (3)

where \(\tilde{u}_n\) is the modified version of vector \(u_n\). Note that deinterpolation yields a vector as its result.

3 Modeling of Finger Holes

In traditional transmission line models for woodwind instruments a side hole is presented as a T network (Keeffe, 1982). This kind of model includes both shunt and series impedances. In practice the series impedance has a small value and can be neglected (Colman, 1979).

It is most important to simulate accurately the open finger holes since they determine the fundamental pitch of the tone. Closed holes have considerably less significant effect and for simplification they are omitted in our model. In the following a computational model for an open hole is derived.

3.1 Model for an Open Finger Hole

The simplest approximation for the acoustic impedance \(Z_0\) of a small open hole is pure acoustic inductance, that is (see, e.g., [Flechter and Rosing, 1991])

$$Z_0(a) = \frac{j \omega \rho c}{A_e} \hspace{1cm} (4)$$

where \(\rho\) is the density of air, \(t\) the effective height of the tone hole (see [Keeffe, 1982] and [Keeffe, 1990]), and \(A_e\) the cross-sectional area.

Multiplication by \(j\omega\) in the frequency domain corresponds to differentiation in the time domain. In the computational model the impedance of an open hole can be represented by means of a digital differentiator \(H_D(x)\). The transfer function of a digital filter approximating Eq. (4) is then

$$Z_0(x) = \frac{j \omega \rho c}{A_e} H_D(x) \hspace{1cm} (5)$$

The equations for modeling a three-port junction are derived in the Appendix of this paper. When Eq. (5) is substituted into Eq. (AB) (see Appendix) the digital reflection function associated with the finger hole junction is given by

$$\hat{R}(x) = \frac{Z_0}{Z_0 + 2Z_0} \left( \frac{\rho c}{A_e} \right) \hspace{1cm} (6)$$

where \(A_e\) is the cross-sectional area of the bore.

The simplest transfer function of a digital filter approximating the differentiator is

$$H_D(x) = \frac{1}{2} (1-x^{-1}) \hspace{1cm} (7)$$

where \(T\) is the sampling interval. This filter computes the difference of two successive input samples and

acts as a high-pass filter. Using this approximation Eq. (6) can be simplified into the form

$$\hat{R}(x) = 1 + a_x x^{-1}$$ \hspace{1cm} (4a)

with the coefficient

$$a_x = \frac{2 \alpha_x T}{2 \alpha_x T + A_e T}$$ \hspace{1cm} (8)

Equation (8a) is the transfer function of a first-order all-pole filter. It resembles a so-called leaky integrator except that it also changes the sign of the signal. Since the coefficient \(a_x\) is always negative this is a low-pass filter. At very low frequencies it acts like a constant multiplier \(-1\).

The magnitude responses of the analog reflection function \(R(x)\) compared with Eq. (4) and the digital reflection function \(R(x)^{\text{eq}}\) of Eq. (8) are illustrated in Fig. 2. The upper curves are computed for a hole radius of 8.0 mm and an effective height of 17.5 mm, and the lower curves for a bore radius of 4.25 mm and an effective height of 31.7 mm (Colman, 1979). The bore radius is 0.95 mm in both cases. The approximation error in the magnitude response of the digital filter is less than 1 dB at frequencies below 10 kHz.

As shown in the Appendix of this paper each finger hole model requires one filter consistent with Eq. (8). By controlling the filter coefficient \(a_x\) in the range \((a_x, 0)\) in a suitable way, the finger hole can be closed and opened as in real woodwind instruments to obtain desired changes in the effective length of the bore.

3.2 Discrete-Time Scattering Junction

Let us consider the implementation of a finger hole model via a three-port junction as shown in Fig. A1. The detailed derivation is presented in the Appendix. The frequency-domain equations (A10) that describe the three-port junction for volume velocity waves can be expressed in the (discrete) time domain as

\[ S \]
In theory, ideal interpolation and deinterpolation cancel each other out when both are applied to a certain signal. Practical Lagrange interpolators and deinterpolators, however, introduce an approximation error which appears as a low-pass filtering effect and phase distortion. Taking into consideration this theoretical background it can be concluded that the terms $h^m u^1$ and $h^m u^2$ in Eqs. (13) introduce an error to the computation and when they are replaced by $u^1$ and $u^2$ this error vanishes. Then Eqs. (13) can be rewritten as

$$\tilde{u}_1 = u^1 + h^m u^2(\alpha) + w(\alpha)$$  \hspace{1cm} (14a)

$$\tilde{u}_2 = u^2 + h^m u^1(\alpha) + w(\alpha)$$  \hspace{1cm} (14b)

Note that these forms only involve deinterpolation. Furthermore, since the last term of the above equations is the same it is economical to compute it separately and then use it in both equations. This yields

$$\tilde{u}_1 = u^1 + \tilde{v}_2$$  \hspace{1cm} (15a)

$$\tilde{u}_2 = u^2 + \tilde{v}_1$$  \hspace{1cm} (15b)

where we have defined a vector

$$\tilde{v}_i = h^m u^j(\alpha) + w(\alpha)$$  \hspace{1cm} (15c)

The signal $w(\alpha)$ is the filtered version of $w(\alpha)$ which also involves fractional indices, and then Eq. (11) has to be rewritten as

$$w(\alpha) = h^m u^1 + h^m u^2 + u^1(\alpha)$$

$$= h^m (u^1 + u^2) + u^1(\alpha)$$  \hspace{1cm} (16)

This form suggests an efficient way to calculate the input for the reflection filter $\tilde{R}(\alpha)$: first add the values in the two delay lines of the digital waveguide and only after that interpolate. Note that as the signals $u^1(\alpha)$ and $u^2(\alpha)$ propagate in opposite directions the coefficients of the interpolating filter appear in time-reversed order with respect to one of them. The computational flow graph of the fractional delay three-port junction is illustrated in Fig. 4.

3.4 The Approximation Error due to FD Filters

Figure 4 shows that the approximation error in the interpolating coefficients $h$ is in effect applied twice to the fractional delay three-port: first in the interpolation of the delay-line value and then in the deinterpolation of the result back to the waveguide. It is

$$\tilde{u}_1 = h^m u^2 + u^1(\alpha) + w(\alpha)$$

$$\tilde{u}_2 = h^m u^1 + u^2(\alpha) + w(\alpha)$$

Fig. 4 A computational flow diagram of a fractional delay three-port junction.
worth pointing out that the interpolated-signal values are only used in the computation of the effect of the junction and deinterpolation is only applied to the result of this calculation. Thus, the interpolation error does not affect the signal that would anyway propagate in the waveguide if the junction were not there.

Fig. 5 shows a high-level block diagram of the finger hole model. The transfer functions $E_i(n)$ and $E_o(n)$ represent the errors of the interpolating and deinterpolating filter, respectively. In this diagram the waveguide is considered ideal, i.e., no errors due to digital implementation are present.

The effect of PD approximation in the finger hole model derived above can be studied by computing the transfer function of a simple bore model. In this experiment one end of the tube is immersed to be ideally closed (i.e., reflection coefficients is 1) and the other end is assumed to be matched (or equivalently an infinite-length tube is considered). A finger hole is placed at a distance $K = 50.5$ unit samples from the closed end. The transfer function of this system is then

$$H(e^{j\theta}) = \frac{2\sin(K\phi) e^{jK\phi}}{1 - E_i(e^{j\omega})H_e(e^{j\omega})E_o(e^{j\omega})e^{-j\theta}}$$

where the transfer functions presenting the interpolation and deinterpolation error are

$$E_i(e^{j\omega}) = e^{j\omega T_i}H_i(e^{j\omega})$$

$$E_o(e^{j\omega}) = e^{j\omega T_o}H_o(e^{j\omega})$$

Figure 6 shows a) the magnitude of the frequency response for the ideal analog system, b) a system with digital reflection filter but with ideal delay lines, and c) a realistic digital system with a digital reflection function and an PD interpolator and deinterpolator implemented by first-order Lagrange interpolation (linear interpolator). It is seen that in this worst case the resonances of the tube are only slightly disturbed in the frequency band of interest, i.e., below 10 kHz.

3.5 Implementation Issues

A single model for an open finger hole is computationally relatively efficient. In practice woodwind instruments have several, up to approximately 20 finger holes. If all of them were simulated simultaneously, the technique introduced in Subsection 3.3, a compiled and computationally heavy model would result. Fortunately, it is not necessary to implement all the open holes, since only the two or three first open holes determine the fundamental pitch of the instrument [Benade, 1960].

The closed holes have slight influence on the effective length of the bore and they also act as low-pass filters. These effects can be taken into account by the simpler methods that by using a fractional delay three-port. Thus, it is suggested that a woodwind bore model with finger holes be implemented based on two or three PD junctions that model the first open holes. When a hole is closed the junction is removed and it is "grown" to another location in the waveguide. Using this strategy the typical playing techniques of woodwind instruments, including the key claps, can be simulated. More tone holes are probably needed for malletophones.

4 A Lossy Tube

In real tubes the heat conduction and viscous drag cause losses. These losses take place on a thin boundary layer near the tube wall.

4.1 Properties of a Lossy Tube

The digital waveguides discussed above are lossless, i.e., they approximate the ideal frequency-domain transfer function

$$H_{\text{fd}}(\omega) = e^{-j\omega D}$$

where $D$ is the propagation delay in unit samples in a tube of the length $L_b$, i.e.,

$$D = \frac{L_b}{v_T}$$

where $v$ is the phase velocity in the tube and $T$ is the sample interval in seconds. The magnitude response

$$|H_{\text{fd}}(\omega)| = 1$$

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of Eq. (19) is unity regardless of the tube length. In real tubes, however, viscous and thermal effects at the walls cause losses which damp the propagating waves as discussed, e.g., in [Benade, 1968] and [Fletcher and Rossing, 1991]. These losses can be accounted for by replacing Eq. (19) by
\[ H_{0}^{2}(\omega) = \epsilon^{2} \rho \frac{a}{\rho_{w} T_{w}} \left( \frac{\omega}{2\pi} \right)^{2} \]  
where the attenuation coefficient \( \alpha(\omega) \) and the phase velocity \( v(\omega) \) are functions of angular frequency \( \omega \):
\[ \alpha(\omega) = k_{0} \sqrt{\omega^{2}} \]  
\[ v(\omega) = c \left( 1 - \frac{k_{0}}{\sqrt{\omega^{2}}} \right) \]  
with
\[ k_{0} = \frac{1}{a} \times 3.0 \times 10^{-3} \]  
\[ k_{0} = \frac{1}{a} \times 1.65 \times 10^{-3} \]  
where \( c \) is the free-air sound velocity and \( a \) is the tube radius in meters. Equations (22a) and (22b) are adapted from [Fletcher and Rossing, 1991] which are modifications from Benade's [1968] formulas. These formulas apply for relatively wide tubes with rigid walls which include those occurring in practical woodwind instruments.

Velocity \( v(\omega) \) in Eq. (22b) differs very little from the free-air velocity in practice. For example, in a bore with a radius of 1 cm the parameter value is \( k_{0} = 0.165 \) which means that \( v(\omega) \) is more than 10% only for \( \omega \geq 3 \) Hz. At practical frequencies (\( \omega > 50 \text{ Hz} \)) the difference between \( v(\omega) \) and \( c \) is less than 3%. Hence, we can safely neglect the frequency-dependence of \( v(\omega) \) and use the constant velocity \( c \).

4.2 Digital Model for a Lossy Tube

It is most straightforward to approximate the frequency response of a lossy tube, Eq. (21), by an FIR filter using the general least-squares approximation method in the specified frequency band. This is done by minimizing
\[ E = \frac{1}{\pi} \int_{0}^{\omega_{0}} \left| H_{L}(\omega) - H_{0}^{2}(\omega) \right|^{2} d\omega \]  
where
\[ H_{L}(\omega_{0}) = \sum_{n=0}^{N} b_{n} e^{-j\omega_{0} \lambda n} \]  
is the frequency response of the \( N \)-th-order FIR filter to be designed and \( \beta \) defines the frequency band of approximation \( 0 < \beta 1 \). The solution can be expressed as [Laakso et al., 1993]
\[ b_{n} = S^{-n} \]  
where \( s \) is the coefficient vector of the FIR filter and \( S \) is an \( (N+1) \times (N+1) \) Toeplitz matrix with the elements
\[ S_{ij} = \frac{1}{\pi} \int_{0}^{\omega_{0}} \cos((k-l)\omega) d\omega, k,l = 1,2, \ldots, N+1 \]  
and \( s \) is an \( (N+1) \times 1 \) column vector with the elements
\[ s_{k} = \frac{1}{\pi} \int_{0}^{\omega_{0}} \left[ R_{k}(\omega) - I_{1}(\omega) \right] d\omega, k = 1,2, \ldots, N+1 \]  
with
\[ R_{k}(\omega) = \text{Re} \left[ H_{0}^{2}(\omega) \right] \cos(k-\omega) \]  
and
\[ I_{1}(\omega) = \text{Im} \left[ H_{0}^{2}(\omega) \right] \sin(k-\omega) \]  
When the sampling frequency is \( f_{s} = 44.1 \text{ kHz} \), the unit delay corresponds to a 7.8 mm long tube (i.e., about 128 unit delays correspond to 1 meter). This means that rather short filters must be used, as the segments between tone holes in woodwind instruments are of the order 2-10 cm, and the approximation precision cannot be increased much if the filter order \( N > 30 \) [Laakso et al., 1993].

The magnitude responses [Eq. (21)] of lossy tubes of the length 5 and 5 cm are plotted by the solid line in Fig. 7 (a = 1.0 cm). The corresponding approximations with 64-p FIR filters are shown by the dashed line. The approximation bandwidth is 11 kHz, that is, half the Nyquist band. It is seen that the approximations are very accurate at low frequencies. (Note that the total vertical scale in Fig. 7 is one dB only.) However, the approximation error varies greatly as a function of the fractional part of the delay (\( D \)) to be approximated.

4.3 Including Losses in the Bore Model

The inclusion of the loss FIR filter of Eq. (26) in the bore model presented in Section 3 is not trivial. The straightforward solution would require a loss filter
between each was delay of the digital waveguide (Smith, 1993), which would be far too laborious and complicated to implement.

A more practical approach is to divide the bore model into segments of delay lines connected by three-port junctions that model the finger holes. A sample in the delay line propagates then in a lossless manner as usual, but its output from the delay line (either to the next segment or to the three-port junction) is corrected by an appropriate loss factor. Note that this kind of lumping of losses does not cause any error in a linear system (Smith, 1993).

The losses are most easily handled with the interpolation filter, i.e., the (lossless) interpolating filters are replaced by their lossy counterparts. Note, however, that the interpolation must remain lossless to avoid doubling the effect.

A more detailed description of the lossy bore model with finger holes will be presented in a future publication.

5 Summary

A method for modeling the open finger holes of a woodwind instrument via three-port junctions was presented. The reflection function of the open hole was approximated using a simple digital filter. It was shown how the location of the side hole in a digital waveguide model can be arbitrarily adjusted by means of fractional delay filters. Furthermore, an FIR filter was designed to simulate the viscous and thermal losses of the bore.

The introduced fractional delay three-port junction is also applicable to waveguide models of musical instruments other than woodwinds. Touching a vibrating string with a finger is an example of a situation which can be simulated using this model structure. Another field of application is the modeling of percussion instruments involving thin bars, like the marimba and the xylophone.

Appendix. Derivation of the Side Branch Junction

The equations governing the reflection and transmission of waves at a junction, where a side branch is connected to a uniform tube, are derived. Generally, for a passive M-way junction, the net volume velocity u vanishes, i.e.,

\[ u = \sum_{m=1}^{M} u_m = \sum_{m=1}^{M} u_m = 0 \]  

(A1)

that is, no net flow from or to the junction takes place.

The pressure across the junction must be continuous. This rule gives another equation which is needed in the derivation of the three-port junction:

\[ p_0 = p_1 = \ldots = p_m \]  

(A2)

Equations (A1) and (A2) are analogous to Kirchhoff's current and voltage laws, respectively, used for electrical networks.

In the following the three-port junction for the acoustic volume velocity is derived. The quantities are illustrated in Fig. A1. Components of the volume velocity are denoted by \( u \) and sound pressure by \( p \). Superscript 'o' stands for the signal propagating forward or towards the end (output) of the bore, and \( 'r' \) denotes the backward direction. Subscript k refers to the index of a piece of the bore and \( \tau \) to the side branch.

The pressure in each branch of the junction at point \( x \) can be expressed in the following form:

\[ p_k(x, \tau) = Z_0 \left[ u_k(x - \tau) + u_k(x + \tau) \right] \]  

(A3)

where \( Z_0 \) is the characteristic impedance of the bore, defined as \( Z_0 = A_0 \), where \( A_0 \) is the cross-sectional area of the bore and \( \tau = \pm c \) is the propagation delay.

The impedance \( Z_0 \) is assumed to be real. However, the impedance \( Z_0 \) of the side hole is complex, i.e., frequency-dependent. Thus, it is necessary to use the frequency-domain equivalent of all the equations in the sequel.

According to Eq. (A2), the pressure signals in the branches of the junction are related in the frequency domain as

\[ Z_0 (U_0^r + U_0^r) = Z_0 (U_0^m + U_0^m) \rightarrow Z_0 (U_0^r + U_0^r) \]  

(A4)

The frequency arguments of the Fourier transforms \( U_0^r \) and \( Z_0 \) in Eq. (A4) have been left out for simplicity of notation.

The volume velocity at point \( x \) inside the digital waveguide is

\[ u_k(x, \tau) = u_k(x - \tau) - u_k(x + \tau) \]  

(A5)

that is, the difference of the components of a volume velocity signal propagating in the delay lines, whereas the total pressure would be the sum of the corresponding components. This is due to the vector nature of the volume velocity.

Substituting the volume velocities, according to Eq. (A5), into Eq. (A1) yields

\[ U_0^r - U_0^m + U_0^m - U_0^r = 0 \]  

(A6)
The output volume velocity signals $U_i^*$, $U_{i+1}^*$, and $U_{i+2}^*$ can now be solved in terms of the input signals $U_i$, $U_{i+1}$, and $U_{i+2}$:

\[
U_i^* = -\frac{Z_0}{Z_4 + 2Z_2(a)} U_i + \frac{Z_0}{Z_4 + 2Z_2(a)} U_{i+1}^* \quad (A7a)
\]

\[
U_{i+1}^* = \frac{-Z_0}{Z_4 + 2Z_2(a)} U_{i+1} + \frac{Z_0}{Z_4 + 2Z_2(a)} U_{i+2}^* \quad (A7b)
\]

\[
U_{i+2}^* = \frac{2Z_0 U_i^* + U_{i+1}^*}{Z_4 + 2Z_2(a)} - \frac{Z_0}{Z_4 + 2Z_2(a)} U_i \quad (A7c)
\]

We define the reflection function as

\[
R(a) = \frac{Z_0}{Z_4 + 2Z_2(a)} \quad (A8)
\]

The equation represents the transfer function between an incident and a reflected signal that propagates in the bore (with impedance $Z_0$). Using this reflection function the Eqs. (A7) can be rewritten as

\[
U_i^* = RU_i^* + (1 + R)U_{i+1} + U_{i+2}^* \quad (A9a)
\]

\[
U_{i+1}^* = RU_{i+1} + (1 + R)U_{i+2} + U_i^* \quad (A9b)
\]

\[
U_{i+2}^* = -RU_i^* + U_{i+1}^* - (1 + 2R)U_{i+2} \quad (A9c)
\]

The function for the volume velocity can be implemented more economically by regrouping the terms of Eqs. (A9):

\[
U_i^* = U_{i+1} + U_i^* + W \quad (A10a)
\]

\[
U_{i+1}^* = U_i^* + U_{i+2}^* + W \quad (A10b)
\]

\[
U_{i+2}^* = U_{i+2}^* - 2W \quad (A10c)
\]

Where

\[
W = R(U_i^* + U_{i+1}^* + U_{i+2}^*) \quad (A10d)
\]

Since the term $W$ appears in each of the Eqs. (A10a)-(A10d), it needs to be computed only once. Thus only one filter $R(a)$ is needed to implement the side branch. An equivalent efficient version of the side hole model for the volume velocity—with the signal $a_i^*$ assumed to be zero—was presented in a condensed form in [Borin et al., 1990].

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References


