THE HARMONIC MATRIX: EXPLORING THE GEOMETRY OF PITCH

Daryn Bond

Winnipeg, Manitoba, Canada
bond@bondinstitute.net

ABSTRACT

Harmonic matrix theory is a comprehensive synthesis of just intonation and Pythagorean tuning systems related to, but extending the classical lambdoma of the ancient Greeks. Extrapolated to n-dimensional structures, cubes, hyper-planes, hyper-cubes, and so on, the approach is completed by the logical inclusion of additional special matrices such as the ‘iterative power’ and ‘golden’ matrix. The harmonic matrix describes a geometry that exposes the fractal nature of the integer set and the significance of the prime numbers to the human perception of harmony.

Applications include compositions not restricted to fixed-tone systems and traditional time structures, various instrument/interface adaptations, and audio-visual representations demonstrating a well-defined link between position, color and pitch.

1. INTRODUCTION

In any measurement system, it is necessary to begin with a fixed point from which comparisons are made. Musical pitch is measured in cycles with respect to the measurement of time in seconds (Hz). The study of harmony is the study of relationships, ratios and proportions to a given reference point, 1/1. (For consistency in this paper, when referring to pitch, 1/1=293.665 Hz – D below A-440, MIDI note no. 62.)

The complexities of creating music using just intonation and Pythagorean tuning systems have confounded composers and instrument builders for centuries [6]. The adequate compromise found in the 12-tone equal-temperament system has solved so many problems that most would look no further [4]. The subject would be closed if only justly intoned intervals and chords did not sound so beautiful [2][3][5].

Contrasting tastes, tuning systems, scales and instrumental designs is not the focus of this paper. The complex subject of human perception, limits of detection, noticeable differences in sensory input, etc. are put aside in favour of computational precision. Limitations with respect to acoustic instruments, notation, execution and performance need not apply in computer music and will not be addressed. A personal computer running any multi-media programming environment can be configured to produce precisely tuned pitch and exact timing. This removes the difficulties confronted by previous generations and allows us to approach the subject anew.

2. DEFINITIONS

2.1. The lambdoma

The classical lambdoma as described by the ancient Greeks is attributed to Pythagoras [7][3]:

\[
\begin{align*}
1/1 \\
1/2 2/1 \\
1/3 2/2 3/1 \\
1/4 2/3 3/2 4/1 \\
1/5 2/4 3/3 4/2 5/1 & \ldots
\end{align*}
\]

All possible whole number ratios are eventually contained within this arrangement. Notice the harmonic series \(P=\{1, 2, 3, \ldots\}\) along the right edge, and the sub-harmonic series \(M=\{1/1, 1/2, 1/3, \ldots\}\) on the left [7][3].

Another way to formulate this is as a matrix (M*P), values at each position \((x,y)\) given by:

\[
x, y \in \mathbb{N} \\
L_{(x,y)} = x / y = xy^{-1}
\]

The harmonic and sub-harmonic series can be joined algebraically, to produce a new series, \(S:\)

\[
x \in \mathbb{Z} \neq 0 \\
S_{(x)} = \left[x^{(x\lfloor x/\lfloor 0)}
\]

Here the exponent unites the harmonic and sub-harmonic series allowing all of the essential information of the lambdoma to be contained in a single term.

2.2. The harmonic matrix

Using this notation, the harmonic plane is defined as (S*S), with values at \((x,y)\) given by:

\[
x, y \in \mathbb{Z} \neq 0 \\
H_{(x,y)} = \left|x^{(x\lfloor x/\lfloor 0)} \ast y^{(y\lfloor y/\lfloor 0)}
\]

This basic formula can be extended to form a 3-dimensional [8-fold] cube by adding a ‘z’ variable. Addition of a 4th variable creates a [16-fold] hyper-plane, and so on.
Values contained within the matrix reach extreme values. For example, a ±16*16 matrix of pitch contains a frequency range from 1/256 to 256 times the fundamental (1.147 Hz to 75,178.18 Hz – well below and above the audible range [1]). To manage this, take the logarithm base 2 of a ratio written as a decimal ‘a0.a1a2a3…’. The integer ‘a0’, corresponds to the octave displacement of the ratio, while the fractional part ‘a1a2a3…’ is the relative pitch class. Replace ‘a0’ with the desired octave and convert back to a ratio. Equation (4) applied to ratio ‘r’ [or to an entire matrix] accomplishes the above in one step.

$$r' = 2^{(\log_2(r) - \lfloor \log_2(r) \rfloor)} \quad (4)$$

At times it may be convenient to use other logarithmic bases for transposition. Base 4 or 8, for example, preserve pitch class and allow for a range of 2 and 3 octaves respectively. Base 3 will transpose by a perfect 12th, base 5 transposes by thirds, etc. A more general procedure can be created using conditional statements to transpose pitch classes to desired octaves. In figure 3, ratios have been constrained to a range of one octave [1,2] and normalized.

Complete analysis of this matrix exceeds available space. A few features of note:

1) All values of the matrix when multiplied together = 1.

2) Applying music terminology, the positive direction is dominant and major, D7(9), and the negative decidedly minor, Gm(6). An interlocking set of perfect triads exists in each of the 4 quadrants.

3) In the upper right and lower left quadrants of the matrix are 2 new sub-matrices, (P*P) and (M*M) [or P^2, M^2]. Notice the P^2 matrix is none other than the familiar ‘multiplication table’ and M^2 its inversion.

4) In the upper left and lower right quadrants of the matrix are 2 lambdomae. A simple variant of this matrix is one consisting of 4 connected lambdomae, given by |x|*|y|⁻¹ (not depicted).

5) Within the lambdoma, every ratio r (0<r<∞), is associated with an angle, θ (5) [see Fig. 4][3].

$$\theta = \arctan(r) \quad (5)$$

6) The extreme right edge of the lambdoma forms an analog of the physical string [Fig. 4]. If ‘q’ is a number of equal divisions of a string (matrix size), then pitches along the string are given by:

$$T = \{q/q, q/q-1, \ldots, q/2, q/1\} \quad (6)$$
7) This series (6) can be connected with its inversion to form a so-called ‘universal string’, or U-string.

\[ U = \{1/q, 2/q, \ldots, q-1/q, q/q, q/q-1, \ldots, q/2, q/1\} \quad (7) \]

2.3. The U-string matrix

The above U series, when transposed by itself \( U(x)^*U(y) \) and constrained, creates a new matrix, the U-String Matrix (Fig. 5).

2.4. The golden matrix

The golden matrix is useful when considering musical form, dividing time or portions of a whole. It is constructed by this simple equation:

\[ G_{(x,y)} = x / (x + y) \quad (8) \]

Though named after the golden ratio \( \frac{x+y}{x} \text{ as } x:y \), \( \phi \) does not appear in this matrix, just as \( \sqrt{2} \) does not appear in the Harmonic Matrix.

2.5. The iterative power matrix [Pythagorean]

This matrix, dedicated to Pythagoras who used \( (3/2)^n \) as the basis for creating musical scales [7], goes a step further by exhaustively transposing all harmonic ratios by one another, \( \delta^3 \).

\[ x, y \in \mathbb{Z} \neq 0 \]

\[ I_{(x,y)} = \left( \left| x \right| \left| y \right| \right)^{\left( x+y \right)/2} \quad (9) \]

![Figure 5. U-string matrix; q=±16, octave [1, 2).](Image)

Observations about the iterative power matrix (Fig. 7):

1) The octave identity results in rows \( \pm1, 2, 4, 8, 16 \ldots \) being shaded uniformly black.

2) The outlined cell in position \((2, -2)\) is the tri-tone, diabolus in musica, \( \sqrt{2} \).

3) On the positive (right) side of figure 7, the row marked ‘3’ represents the cycle of perfect fifths, and ‘-3’ the cycle of perfect fourths.

4) Zoom out on the situation, and the fractal structure of harmonic relationships becomes apparent (Fig. 8).

![Figure 6. The golden matrix; n=16.](Image)

![Figure 7. Pythagorean Matrix \( n=±16 \), octave [1, 2).](Image)

Implications of this theory are far reaching. When utilizing harmonic matrices, it is not sufficient to talk of modulating from ‘D’ to ‘C’. To be precise, a composer would indicate a modulation from \( 1/1 \) to \( 16/9 \), or to \( 7/4 \), or by an equally tempered factor of \( 10^{12}\sqrt{2} \), and from there precisely indicate the desired sonority.
Matrices need not be square, rows and columns can be filtered out or repeated. These equations can be manipulated to create an infinite variety of fractal landscapes and harmonic surfaces.

Though this discussion has primarily used pitch as example, these matrices are applicable to any system of proportional measurement and can be similarly used to analyze, organize and realize time structures.

![Figure 9](image)

**Figure 9.** equation (2) $S_{kl}$ applied to rhythm.

### 3. ACCESS AND APPLICATIONS

The development of this approach to harmony has occurred in tandem with instrumental adaptations, compositional approaches and graphical representations. The infinite number of available pitches and rhythms make these innovations necessary. It is important to emphasize that these matrices are not just mathematical abstractions, but must be heard to be fully appreciated.

#### 3.1. Instruments and interfaces

Virtually any electronic input or data stream can be mapped to access the harmonic matrix. Any sound-producing module capable of accepting precise values can be used. The equations in this paper present flexible and intuitive mapping strategies. Examples of existing technologies that have been adapted to access the harmonic matrix include sliders, dial, the mouse, computer & MIDI keyboards, graphics tablets, video camera, various sensors, 3-d accelerometers [e.g. wiiremote] and multi-touch surfaces. Connected to a computer, these interfaces can be dynamically reconfigured to account for transpositions or changes to matrix size and structure.

#### 3.2. Algorithmic composition

Data generated algorithmically, using serial, minimal, deterministic or stochastic compositional techniques, is used to explore the possibilities of the matrix. Parameters of pitch, rhythm, timbre, velocity, panning, and effects can be generated and used independently or in conjunction with instrumental interfaces, opening the way for a music that goes beyond fixed pitch systems and basic time structures.

#### 3.3. Representations

Graphic representations of harmonic matrix data provide useful visual feedback. The visualization of matrix data can be drawn with geometric precision while the music is being created. Thus far, preliminary 3-dimensional renderings of harmonic cube data have been implemented. Input data is used to calculate pitch and position. Colorizing the matrix makes use of the fact that the human eye sees somewhat less than one octave of light – from 390 to 750 nm [1]. In figure 10, the visible spectrum is ‘artfully’ extended from 373 to 746 nm. Pitch class is associated with a color, octave with brightness, and a path through the resulting cube is rendered in openGL space.

![Figure 10](image)

**Figure 10.** colorized input data rendered in 3-d space.

### 4. SUMMARY

Study of the harmonic matrix presents the subject of rational relationships in a new light. Simple mathematics reveal a fertile new territory set upon a very old ground – one that could not until now be fully explored.

Applications of the matrix go beyond creating intuitive, dynamic, harmonious musical instruments, or producing complex new compositions, but extend into the field of scientific education and exploration, demonstrating an elegant, fundamental link between mathematics and music.

### 5. REFERENCES


