FRAC TAL DEPTH STRUCTURE OF TONAL HARMONY

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ABSTRACT: The paper presents an algebraic/ geometric approach for the morphological study of Western Harmony. Within the framework of Mathematical Music Theory, chords are studied by means of affine endomorphisms of the 12-tone system, which we call fractal tones. The consideration of fractal chords allows an intentional study of harmonic functions. In order to signify a tonal function, the fractal closure of a given chord must contain significant fractal tones. The classical contrapuntal consonance/dissonance dichotomy of intervals gives rise to a similar(!) dichotomy of fractal tones. Harmonic thinking in just intervals and the 12-tone system are related to each other through an "uncertainty relation" in the symmetry group of the Euler Algebra. We present the hypothesis that the commutation of this symmetry group \textit{SL}_2(\mathbb{Z}) describe the cognitive counterpart of what musicians call commata.

1. Fractal chords and tonal functions. In this first paragraph chords are considered to be simply subsets of the 12-tone system, which is described by the ring $\mathbb{Z}/12\mathbb{Z}$ of integers $\mathbb{Z}$ modulo 12. There are 144 affine endomorphisms of the 12-tone system

$$A = \{[a, b] : T \rightarrow T \{[a, b](t) = a t + b; (a, b \in \mathbb{T})\}$$

which we call fractal tones, since endomorphisms represent self-similarities. The composition of two fractal tones is given by the formula $[a_2, b_2][a_1, b_1]=[a_2a_1, b_2b_1+b_2]$. 

Semigroups of fractal tones are called fractal chords. For any (nonempty) chord $X \subseteq T$ we consider the fractal closure $A(X) = \{X \in A(|\mathbb{A}| X) \subset X\}$. 

This notion makes sense because "ordinary tones" - the elements of $T$ - may be identified with the constant (fractal) tones of type $[0, b]$. Besides the constant tones, the half-constant tones $[b, b]$ and the four types of symmetry tones $[1, b], [1, b], [5, b], [7, b]$ there are two more types of fractal tones, which are essential for the intensional study of tonal functions: minus tones $[a, b]$ with $a = 3, 9$ and plus tones $[a, b]$ with $a = 2, 4, 8, 10$. 

The significance of tonal functions are fractal chords $F = <\alpha, \beta>$ generated by a minus tone $\alpha$ and a plus tone $\beta$. The evaluation of whether a given chord $X$ may signify the tonal function $F$ is characterized by the position of the intersection $A(X) \cap F$ within the Hasse-diagramm of all subsemigroups of $F$ - the intention of $F$. In the case of the Tonic in C-Major we have $\alpha = [3, 1]$ and $\beta = \{8, 4\}$. $F = \{\alpha, \beta\}$ contains three constant tones: $c = [0, 0], g = [0, 1], e = [0, 4]$. The Parallel- and the "Gegenklang" functions correspond to $<\beta> = [\beta, \beta^2]$ and $<\alpha> = [\alpha, \alpha^2]$, the mediant correspond to $\alpha^2$ and $\beta^2$. 

The paradigmatic relation of two chords $X$ and $Y$ may be studied by considering the set $A(X, Y) = \{X \in A(|\mathbb{A}| Y) \subset X\}$ of all fractal tones mapping $Y$ into $X$. In the case of the "fifth-fall" of any major chord $X$ (say $X = (1, 2, 5), Y = (0, 1, 4)$ this set $A(X, Y)$ generates fractal chord $KONS = [a, b] \in A(|\mathbb{A}| \alpha = \{0, 1, 3, 4, 8, 9\}$) of 72- elements, which we call the consonant fractal tone. As a surprising fact one should notice, that $(0, 1, 3, 4, 8, 9)$ represent the consonant intervals according to the Fuxian dichotomy (cf. Mazzola 1990).
2. Enharmonicity. The description of harmonic thinking in just octaves, fifths and thirds is based on the study of a 5-dimensional lattist E spanned by $\alpha = \ln 2$, $\beta = \ln 3$ and $\gamma = \ln 5$ over the ring $Z$ of integers - the Euler module. Within E one has the commutative sublattice $K$, spanned by $\alpha$, $\beta = 12\alpha$ and $\gamma = 4\alpha - 4\beta + \gamma$. Enharmonic and octave identification is then described by the projection map enh: $E \rightarrow E/K$, where $E/K$ is a finite cyclic lattice of order 12. Since E is a dense subset of the reals $\mathbb{R}$, the traditional descriptive or normative characterisation of commata as acoustic compromises lacks explanatory power. The following considerations suggest a plausible way to explain the phenomenon of enharmonicity.

With respect to the basis $\alpha$, $\beta$ and $\gamma = 4\alpha - 5\beta + \gamma$ we identify the Euler module $E$ with the additive structure of the special linear algebra $SL_2(Z) = \left\{ \begin{bmatrix} k & m \\ n & k \end{bmatrix} | k, m, n \in Z \right\}$

by setting: $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\gamma = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

The Lie bracket $[\alpha, \beta] := \alpha \beta - \beta \alpha$ then also induces a Lie algebra structure on E, which then shall be called Euler algebra. Let $\mathcal{F}$ denote the sublattice of $E$ spanned by $\alpha$ and $\gamma$.

The projection $proj: E \rightarrow \mathcal{F}$ corresponds to the identification of octave classes. On $\mathcal{F}$ we define the exponential function $exp: \mathcal{F} \rightarrow SL_2(Z)$ (Special Linear Group $SL_2(Z)$)

$exp \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $exp \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}

For arbitrary $x = \alpha \gamma + \gamma \alpha$ we set $exp(x) := exp(\alpha)exp(\gamma)$. The elements of $G = SL_2(Z)$ may be interpreted as inner symmetries of the Euler algebra. This symmetry group $G$ is non commutative. It's commutator subgroup $[G,G]$ generated by the commutators $[\alpha, \beta]$ and $[\gamma, \beta]$ is of index 12 (cf. Noll 1994: 119ff). Via the exponential function commata (and only commata) correspond to commutators of $G$. Hence the abstract 12-tone system appears as the homology group of $G$, i.e. as a group of symmetry classes of the Euler algebra due to the uncertainty relation of non commutative symmetries. The latter somehow may be related to the processuality of cognitive operations.

3. Remark: The implementation of this theory within the Zürich RUBATO®-project (cf. Mazzola 1993/94) will provide a powerful tool for the experimental study of harmony in relation to performance.

4. References:
Mazzola, Guerino (1993/94), Geometry and Logic of Musical Performance III. University of Zürich.