Abstract

The distributed reflection of the flared tube is calculated using the Schroedinger equation, including the effects of viscothermal and open end radiation losses. The reflection is approximated with digital filters, which are designed to be efficient and accurate in a digital waveguide model of a wind instrument. Weighted Least-Squares design techniques are applied to model the reflectance of a Bessel horn with an IIR filter.

1 Introduction

Digital Waveguide Filters (WGF) are widely used for simulation of wave propagation in musical instruments. The particular efficiency of WGF implementations is based on the separation between unfiltered propagation (digitally represented with delay lines), and additional physical phenomena, such as reflections, losses and dispersion (approximated with digital filters).

WGF models have been successfully applied for wind instruments, where the main bore of the instrument is often represented by a single elemental tube shape (cylindrical, conical). Flared tube sections (bell, mouthpiece) are usually approximated with a series of small elemental segments. (see, f.e., [1], [2]).

In the case of modeling a strongly flared tube (such as brass horns), using piecewise cylindrical segments often requires a large number of segments to enable sufficiently accurate results [3]. A much smaller number of conical segments would be enough, but unfortunately the time-domain filter representations of conical junctions have structural inherent instabilities [2]. Alternatively, the complete reflection and transmission characteristics of the horn may be calculated by applying Schroedinger's equation [4], and approximated with a single digital filter. This paper addresses the design of such a filter in a WGF Model.

In section 2, the use of Schroedinger's equation is discussed, and a formulation with scattering and propagation matrices is presented. Further extensions are made in section 3 by incorporating the effects of viscothermal losses and open-end radiation. The design of an appropriate digital filter is discussed in section 4. Under the assumption that frequencies above the horn cut-off are mainly transmitted out of the horn, we aim to obtain a digital filter that has its main discrepancy in the phase response at the higher frequencies. The phase of the highpass transmitted wave energy is considered to be of minor perceptual importance, whereas the phase of the lowpass reflectance is essential to the sound generation mechanism in wind instruments, and should therefore be approximated accurately. The idea of 'guiding' the main discrepancy into the phase of the higher frequencies is to obtain a horn waveguide structure containing low-order digital filters that will be capable of sufficiently accurate real-time simulation.

With these considerations in mind, both FIR and IIR filter design techniques were applied to approximate the analogue filter characteristics. As an example we applied our calculations to a WGF model of a Bessel horn, which is used in a trumpet simulation by Federici and Borin [5].

2 Schroedinger's Equation

Wave propagation in tubes of varying cross-section is modeled by Websters’s equation:

$$\frac{1}{c} \frac{d^2 p}{dt^2} = \frac{\partial^2 p}{\partial z^2} + \frac{1}{S} \frac{dS}{dz} \frac{\partial p}{\partial z}$$

(1)

where $z$ is the coordinate that corresponds to the position where the spherical wavefront surface $S$ cuts the tube axis (see figure 1). In [6] a convenient coordinate transformation is used to rewrite (1) into the Schroedinger equation:

$$\frac{\partial^2 \psi}{\partial z^2} + (k - F_k(z)) \psi = 0$$

(2)
Here $\psi = p\sqrt{S}$ is an alternative wave parameter, $k = \omega/c$ is the free space wave number, and $F_h(z) = \sigma^2 r/\partial z^2$ is the 'horn function'.

![Figure 1: Wave propagation in the flared tube](image)

In order to find the filter characteristics of the wave propagation-path of $\psi$-waves in flared horns, we assume that the horn acts as a one-dimensional waveguide: i.e., $\psi$ is a sum of right- and left-going waves:

$$\psi = \psi^+ + \psi^- = Ae^{-j\beta z} + Be^{j\beta z}$$

(3)

All attenuation and dispersion effects in the horn are expressed in the *locally defined* propagation constant $q(z)$, which is found by substituting (3) into (2):

$$q(z) = \sqrt{k^2 - F_h(z)}$$

(4)

According to (4), the characteristics of lossless wave propagation are spatially varying (as opposed to wave propagation in cylindrical and conical tubes), and are determined by the horn function. Since this function can be of arbitrary shape, no exact solutions of (2) can be found, and spatial discretization of the wave propagation-path is necessary. Assuming that at junctions between tube sections the pressure is constant and the total sum of incoming volume velocity flow equals zero, we found the following *scattering matrices*:

$$P_n = \begin{bmatrix} e^{jq_n} & 0 \\ 0 & e^{-jq_n} \end{bmatrix}, \quad J_n = \begin{bmatrix} 1 + \frac{q_{n+1}}{q_n} & 1 - \frac{q_{n+1}}{q_n} \\ 1 - \frac{q_{n+1}}{q_n} & 1 + \frac{q_{n+1}}{q_n} \end{bmatrix}$$

(5)

where $P_n$ is the propagation matrix that represents the traveling of waves in section $n$ of length $l_n$, and $J_n$ is the scattering matrix for the junction $(n,n+1)$. Note that $P_n$ represents *filtered* wave propagation, and can not be easily simulated in the digital domain. The final matrix $F$ that relates $\psi$-waves on the right side of the tube to $\psi$-waves on the left side, is calculated as a product of $N$ cascading sections:

$$\begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix} = F = J_0 \cdot P_1 \cdot J_1 \cdot \cdots \cdot P_n \cdot J_n \cdot \cdots \cdot P_N \cdot J_N$$

(7)

where $q_0 = q_{N+1} = k$. The horn reflection coefficient is computed as: $R_h(\omega) = G/E$.

### 3 Incorporation of Losses

The effects of viscothermal losses can be included by using a 'lossy' wave number that varies with angular frequency $\omega$:

$$k(\omega) = \omega / \nu(\omega) - j\alpha(\omega)$$

(8)

with sound velocity $\nu(\omega)$ and attenuation factor $\alpha(\omega)$, for which appropriate values can be found in most literature on acoustics (see, e.g., [9]). The losses due to the open-end radiation can be represented with scattering (termination) matrix $T$:

$$T = \frac{1}{1 + R(\omega)} \begin{bmatrix} 1 & -R(\omega) \\ R(\omega) & 1 + 2R(\omega) \end{bmatrix}$$

(9)

A suitable expression for the open-end reflection coefficient $R(\omega)$ is given for spherical wavefronts by Causse et al. [3]. The final EFGH-matrix is now computed by $F \cdot T$.

### 4 Digital Filter Approximation

It is necessary to approximate the reflection function by a low-order digital filter to render it feasible for real-time sound synthesis. It would be easiest to use an IIR filter, but since the impulse response corresponding to the reflectance of a flared tube is usually quite long, a large number of FIR filter coefficients would be needed and this would lead to an inefficient implementation. IIR filters are more complicated to design but they allow effective realization of long impulse responses with a small number of arithmetic operations. An IIR filter design method that allows separate weighting of specified magnitude and phase (group delay) has been developed by Deczey [7]. The method optimizes the poles $(r_p,\phi_{p,i})$ and zeros $(r_o,\phi_{o,i})$ of a cascade of second-order sections:

$$H(z) = k_0 \prod_{i=1}^{N} \frac{1 - 2r_p,\cos(\phi_{p,i})z^{-1} + r_p,^2z^{-2}}{1 - 2r_o,\cos(\phi_{o,i})z^{-1} + r_o,^2z^{-2}}$$

(10)

Iterative gradient-based techniques are applied to minimize a weighted sum of least-$p$ errors over a discrete set of frequencies $\omega_k$. The main disadvantage of this method is that good initial pole positions are required to avoid the algorithm from halting at local minima [8]. In practice, we found that restricting the parameter optimization entirely to either magnitude...
response or group delay strongly simplifies the choice of the initial parameters and improves the performance of the method. Hence a more robust procedure is to approximate the specified magnitude $A_s$ and group delay $\tau_s$ with two separate filters. Once a suitable magnitude approximation filter $H_m$ is found, the desired group delay $\tau_d$ for the second filter becomes:

$$\tau_d(\phi) = \tau_s(\phi) - \tau_m(\phi) \quad (11)$$

where $\phi_k = \omega_k T$, with sampling period $T$. In order to ensure that the second filter does not boost or attenuate any frequencies, (10) is modified into the allpass filter:

$$H_a(z) = \prod_{i=1}^{N} \frac{r_i^2 - 2r_i \cos(\phi_i) z^{-1} + z^{-2}}{1 - 2r_i \cos(\phi_i) z^{-1} + r_i^2 z^{-2}} \quad (12)$$

The following Least-square (LS) error norm was used to determine the coefficients of this filter:

$$L_t = \sum_{k=1}^{K} w_k \left[ \tau_a(\phi_i) + \tau_0 - \tau_d(\phi_i) \right]^2 \quad (13)$$

with weighting function $w_k$. The partial derivatives $\partial L_t / \partial r_i$ and $\partial L_t / \partial \phi_i$ can be expressed in simple closed form and are applied to find the direction towards the minimum error. The stability of the filter is guaranteed by restricting magnitude of each pole $(0 < r_i < 1)$. At each iteration the nominal group delay $\tau_0$ is optimized and forced to be an (even) integer:

$$\tau_0 = \text{even} \left( \sum_{k=1}^{K} w_k \left[ \tau_a(\phi_i) - \tau_d(\phi_i) \right] \right) / \left( \sum_{k=1}^{K} w_k \right) \quad (14)$$

This way the procedure determines the number of delays that might have to be added or subtracted from the delay lines that represent the main bore (see figure 2). The fractionality of the delay is then also incorporated in the allpass filter. To avoid extra filters for the viscous thermal losses of the main bore, these may be included into the bell reflectance computation.

### 5 Results and Implementation

The method was applied to a 54cm long Bessel horn with a bore profile described by:

$$r(x) = b(x + a)^\gamma \quad (15)$$

For $a = 1.05 \text{cm}$, $b = 12.7 \text{cm}$, and $\gamma = 0.6$, the radius $r$ as a function of tube-axis position $x$ forms a good approximation to a trumpet bell. The corresponding horn function is calculated with cubic interpolation [6].

Figure 3 shows the impulse response of the horn. The response decays quite slowly and thus an FIR approximation would not be effective. Three second-order IIR sections were required to approximate the magnitude of the analogue response of the Bessel horn with an average accuracy of 0.25 dB. Here we used equal weighting of a LS error norm in a band of 2,5 KHz, with a sampling frequency of 44.1 KHz (see figure 4a). Using more sections did not improve the accuracy substantially. A larger number of sections ($N = 5$) is needed for the allpass group delay design (see figure 4b). Fortunately the second-order allpass-sections are computationally efficient structures, which can be implemented using just 2 multiplications per section:

$$y(n) = a_{2,1}[x(n) - y(n - 2)] + a_{1,1}[x(n - 1) - y(n - 1)] + x(n - 2) \quad (16)$$

with coefficients $a_{1,1} = -2r_i \cos(\phi_i)$ and $a_{2,1} = r^2$.

The total number of multiplications for this filter is $1 + 3 \times 4 + 5 \times 2 = 22$. This result is comparable to a conical section approximation with similar accuracy. With strategically chosen junction locations, about 10 sections are required for a strongly flared tube [3], and its WGF implementation would have $10 \times 2 = 20$ multiplications, thereby ignoring the extra fractional delay filters [2] that will be needed.

Figure 2: WGF model of wind instrument bore

An LS FIR approximation corresponds to truncating the beginning of the impulse response and using that as the filter approximation. In order to approximate the impulse response of the Bessel horn an FIR filter would have about 300 coefficients corresponding to a duration of 6.8 ms.

Figure 3: Calculated impulse response of the Bessel horn
6 Conclusion

A digital waveguide model of strongly flared acoustic tubes has been developed, including the effects of viscothermal and open-end radiation losses. All these aspects are 'lumped' in our model into two digital filters (one for magnitude and another for phase approximation). The system is designed to have good magnitude characteristics over the entire Nyquist band and good phase properties for those frequencies which are most strongly reflected. Formerly similar models have been realized using cylindrical and conical waveguides and fractional delay filters. The main advantages of the proposed modeling scheme are that (1) the model can be guaranteed to be stable by careful design of the digital filters (whereas conical waveguide filters are unstable in some cases) and (2) separate fractional delay filters that are computationally expensive are not needed. A minor disadvantage is the lack of parametrization since new digital filters need to be designed when the properties of the horn are changed. Fortunately, the geometry of the air column of most musical horns is not changed during playing so the filters can be designed beforehand.

Strongly flared tubes, such as the Bessel horn that we used as an example, have a long impulse response that is more efficiently modeled with IIR filters than FIR filters. By separating the magnitude and phase aspects of the IIR design procedure the phase approximation error can be guided into the less important high frequencies. This approach constitutes a structural improvement of the balance between accuracy, efficiency, and perceptual relevance of WGF models of horns and wind instrument bells.

References


