COUNTING AS A TYPE OF MEASURING

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1. The Challenges

One of our earliest lessons in formalizing natural language is that we can translate counting sentences like ‘Two frogs are in the bog’ into first-order predicate logic with identity. According to the lesson, (1), on the reading in which it expresses an exact count (rather than an at least count), can be formalized as (2).¹

(1) Two frogs are in the bog.

(2) $\exists x \exists y (\text{frog}(x) \land \text{frog}(y) \land x \neq y \land \text{in the bog}(x) \land \text{in the bog}(y) \land \forall z ((\text{frog}(z) \land \text{in the bog}(z)) \rightarrow (z=x \lor z=y)))$

Comfortable as it may be, Salmon (1997) taught us that the familiar translation procedure is problematic when counting is sensitive to partiality. It is puzzling, for instance, how we could use the procedure to formalize (3). After all, we have no symbol for half-existential quantification.

(3) Two and a half bagels are on the table.

Worse still, the translation procedure doesn’t always capture the truth-conditions of counting sentences without fractions. If there are two and a half bagels on the table, then (4) is false, though its translation (5) is true.

(4) (Exactly) two bagels are on the table.²

(5) $\exists x \exists y (\text{bagel}(x) \land \text{bagel}(y) \land x \neq y \land \text{on the table}(x) \land \text{on the table}(y) \land \forall z ((\text{bagel}(z) \land \text{on the table}(z)) \rightarrow (z=x \lor z=y)))$

1. For the remainder of the paper, I’ll work with the exact readings of counting sentences, unless I explicitly state otherwise. Focussing on the exact reading seems reasonable given that Horn (1992), Geurts (2006), Breheny (2008), and Kennedy (2013) argue that exact readings are semantically available rather than pragmatically derived. That said, Liebesman (2015) shows how to extend most of the considerations in this section to at least readings.
2. ‘Exactly’ is in parentheses because it can be omitted as long as context forces an exact reading of the sentence.
My aim is to provide an account of counting, including a semantics for number terms, that accounts for partiality-sensitivity. In the remainder of this section, I’ll outline the key challenges posed by partiality-sensitivity. In sections 2—4 I develop a view on which counting is a type of measuring.

1.1 Challenge 1: The Logic-Class Procedure
I’ve already sketched first challenge: that the familiar formalization procedure fails when sentences contain terms for mixed fractions, or when counting sentences are uttered in contexts in which partiality is relevant.

According to our translation procedure, and focussing on the exact reading, there are only two possible truth-conditional glosses of (3): one contains two existential quantifiers and the other contains three. The problem is that neither of these is truth-conditionally equivalent to (3): the former is true if we remove a half-bagel from the table, and the latter requires an additional half in order to be true.

(3) Two and a half bagels are on the table.

At least prima facie, formalizing sentences like (3) requires resources beyond first-order predicate logic with identity. A natural reaction this reasoning is that it ignores the complexity of ‘two and a half’. I’ll return to this. However, before we get to that, recall that reflection on the way in which half-bagels affect counting also undermines the translation procedure for (4), which contains no such complexity.

(4) (Exactly) two bagels are on the table.

On our familiar translation procedure, the truth-conditions of (4) are given by (5). However, the two are not truth-conditionally equivalent. Recall our initial scenario in which there are two whole bagels and one half-bagel on the table. In this scenario, (4) is false: after all, there are exactly two and a half bagels on the table, not exactly two. (5), however, is true. The two whole bagels witness the truth of the doubly quantified formula, and since half-bagels are not bagels, there is nothing to undermine its truth.

These challenges are often met with skepticism, and I’ll now respond to the most tempting form. Liebesman (2015) contains a more detailed discussion.

In examining (3), I’ve taken ‘two and a half’ to contribute a single numerical value. There’s a worry that this is misguided. Perhaps, the worry goes, when we use (3) to count, we’re providing two different straightforward counts, one that counts the bagels and another that counts the half-bagels. On this view, ‘and’ in ‘two and a half’ conjoins two distinct counts rather than building a single complex number-designating term. If this is correct, then we could hold that the familiar translation procedure need only be slightly modified to deal with (3), but no resources beyond first-order predicate logic with identity are required. This is similar to the proposal made by Ionin and Matushansky (2006), which I discuss in more detail in section 3.3.

The most straightforward problem with this objection is that it fails to account for the falsity of (4). (4) does not contain a number-term expressing a mixed fraction, so it is hard to see how a more complex analysis of ‘two and a half’ in (3) can help us gain any traction on its partiality-sensitivity.

The skepticism also fails when we focus on (3) itself. The skeptic will take (3) to be elliptical for a more complex sentence like (6) or (7).

(6) Two bagels and a half-bagel are on the table.

(7) Two bagels are on the table and a half-bagel is on the table.

These paraphrases, however, fail for at least two reasons. The first is that they fail to account for the ease of anaphoric reference to 2.5. (Here and throughout, I’ll take numerals to refer to numbers.) Consider (8):

(8) Two and a half bagels are on the table. Twice as many onions as that are on the counter.

The truth-conditions of the second sentence in (8) are clear: it is
true just in case five onions are on the counter. To generate these truth-conditions, ‘that’ must designate 2.5. It is hard to see how it would unless ‘two and a half’ make 2.5 available for anaphoric reference (at least without significantly enriching the context). Of course, these sorts of appeal to anaphora are not decisive: we know that there are many ways to make objects available for anaphoric reference beyond designating them. That said, the claim that ‘two and a half’ designates 2.5 would allow a straightforward and plausible explanation of the truth-conditions of (8).

Even setting aside anaphora, (3) cannot be elliptical for (6) or (7) for a simple reason: they don’t have the same truth-conditions. Consider a situation in which I place two whole bagels on the table and then four eighths of another bagel that I’ve carefully divided. In such a case, (3) is true: there are two and a half bagels on the table in virtue of there being two whole bagels and four eighths of another. Neither (6) nor (7) is true in such a case. There is no half-bagel on the table: the only things on the table are whole bagels or bagel-eighths. Again, this isn’t decisive; one could try to find a more complex paraphrase of (3) that sidesteps the problem. However, without providing some independently plausible method for deriving the paraphrase, the objection will fail.

3. This is hardly the end of the dialectic. As David Nicolas (p.c.) insightfully points out, number-referring anaphoric pronouns can sometimes designate the sum of two salient numbers. His example is ‘Two bagels were on the side table, three bialys were in the bread box, and twice as many kaiser rolls as that were on the counter.’ His claim is that there is a reading of this sentence on which ‘that’ refers to 5. I agree that there is such a reading. However, there are three main reasons that I don’t take this to undermine the argument from anaphora. Each of these three reasons show that there are substantial disanalogies between Nicolas’ sentence and (8). First, there is also a reading of Nicolas’ sentence on which ‘that’ refers to 3. However, there is no reading of (8) on which ‘that’ refers to 1. Second, I find it much harder to get the reading of Nicolas’ sentence on which ‘that’ refers to 5 than I find it to get the reading of (8) on which ‘that’ refers to 2.5. Third, if we take seriously (6) or (7) as paraphrases of (3), we’d predict that ‘that’ in (8) would refer to 3, not 5! This is due to the fact that ‘two’ provides 2 and ‘a’ provides 1.

4. One way to see why it is so tempting to hold that the restrictor position is extensional is to consider Hume’s principle. According to Hume’s principle, the number of Fs is equal to the number of Gs if and only if the Fs are equinumerous with the Gs. Most find Hume’s principle plausible, even if there are disagreements about what role it is suited to play in the foundations of mathematics. The most familiar debate concerns whether it is analytic. See Boolos (1997), Wright (1998), and Heck (2000). Some may reject Hume’s principle because they don’t believe in numbers, or for some other theoretical reason. However, if one takes the restrictor position to be extensional, one will reject Hume’s principle for a more straightforward reason. Assuming that ‘F’ and ‘G’ express different intensions, it could be the case that every F is a G, and vice-versa, so that the Fs and Gs are equinumerous, even if the number of Fs (relative to F) is different than the number of Gs (relative to G).

All of this shows that there is no straightforward modification of the familiar translation procedure that yields the correct truth-conditions. This gives rise to the following question.

Q1: What resources do we need to adequately capture the truth-conditions of counting sentences, and how do they go beyond first-order predicate logic with identity?

1.2 Challenge 2: Intensionality in the Restrictor Position

Counting sentences have the form ‘N F are G’, where N is a number term and F and G are general terms. In our logic-class translation procedure, and in familiar account of the semantics of number terms, the F-position — henceforth ‘the restrictor position’ — is extensional. The second challenge is that reflection on partiality-sensitivity seems to show that the F position is intensional.

In order to show that (3) is intensional in the restrictor position, which is occupied by ‘bagels’, we need to find a term co-extensive with ‘bagels’ which cannot be intersubstituted with ‘bagels’ preserving truth. Given the proper context, ‘New York bagels’ is such a term.

(3) Two and a half bagels are on the table

(9) Two and a half New York bagels are on the table.
Here’s a context in which (3) is true but (9) is false, despite the fact that ‘bagel’ and ‘New York bagel’ are co-extensional. I have just travelled from Montreal to New York with a bagel in tow. Excited to test the virtues of Montreal bagels, I eat half of my Montreal bagel, and place the remaining half on the table along with two whole New York bagels from the shop on the corner. (3) is true: there are two and a half bagels on the table. (9), however, is false: the half-bagel is from Montreal, so it doesn’t add to the count of New York bagels. Now for the bad news: we’re also in the middle of a bizarre bagel-blight (a carb-craving evil demon?) and the only whole bagels left in the world are New York bagels. (If that’s too strange for you, just imagine a combination of a flour-shortage and lack of bagel interest outside of NYC.) The bagel-blight ensures that the only things in the extension of ‘bagel’ are New York bagels. ‘Bagel’ and ‘New York bagel’ are, then co-extensional.

Of course, one may doubt that I’ve genuinely described a scenario in which ‘bagels’ and ‘New York bagels’ are co-extensional. Note that though I am free to stipulate most of the details of the scenario, I cannot stipulate that the two terms are co-extensional. This is because an essential component of the scenario is that it contains a non-New York half-bagel. If such half-bagels are themselves in the extension of ‘bagel’, then this essential component will undermine the co-extensionality of ‘bagel’ and ‘New York bagel’.

So, are half-bagels in the extension of ‘bagel’? I don’t have an argument here, but my strong sense is that they aren’t. However, we need not rely on substantial claims about the metaphysics of brunch. We can easily modify the example. Imagine that the relevant sentence is ‘Two and a half bagels are on the table’. The temptation to treat a bagel-eighth as in the extension of ‘bagel’ (in material mode: as a bagel) is, I take it, minimal.

Furthermore, even if one is extremely stubborn, and insists that half-bagels and eighth-bagels are in the extension of ‘bagel’, we can modify the example further. Imagine that we are in a very sparse world that contains nothing but two whole bagels and one half-bagel in a void (no cream cheese!). In that case ‘bagel’ will be co-extensional with ‘maximally continuous thing’: both will (on the view being considered) have the whole-bagels and the half-bagel in their extension. Nonetheless, the two cannot be intersubstituted salva veritate, as exemplified in the following pair:

(10) Two and a half bagels are in the void.
(11) Two and a half maximally continuous things are in the void.

(10) is true while (11) is false. Three things are in the void; two and a half bagels are three things.

As with our first challenge, these intuitions generalize to sentences that do not contain terms for mixed fractions. Compare (12) with (13):

(12) (Exactly) two bagels are on the table.
(13) (Exactly) two New York bagels are on the table.

In the scenario described above, (12) is false, but (13) is true. So, even when the number word in a counting sentence picks out a natural number, intensionality remains.

These observations yield our second question:

Q2: How can we explain the intensionality in the restrictor position of counting sentences?

1.3 Challenge 3: The Frequent Success of the Familiar Procedure

The familiar translation procedure is standard fare in our introductory logic classes, and (I can testify) seems so obvious to many as to not bear discussion. This attraction is not to be discounted. It stems from a simple fact: the procedure delivers truth-conditions that yield the
correct truth-value in many familiar situations. Notice that I haven’t claimed that the truth-conditions are correct. The distinction between truth-conditions yielding the correct truth-value and being correct is an important one; I’ll return to it in 2.3. For now, it suffices to note that there’s something right about the familiar truth-conditions. If we abandon the procedure, we nonetheless need to understand why it is so tempting and successful when partiality is not relevant.

Q3: What accounts for the successes of the familiar translation procedure?

1.4 Challenge 4: The Failure of Unrestricted Summation

Here’s a surprising fact observed by Salmon (1997: 240): counting doesn’t allow us to freely sum partial entities. Consider the following scenario:

The Toyota factory produces Yarises in a two-step process. First, the entire front half is completed, then the entire back half is completed. Around 10 AM on Monday, the factory has completed a front half, in addition to the complete Yaris that is already on the floor. ‘One and a half cars are completed’ is true. It is then discovered that the factory is out of back-bumpers. Instead of constructing any more back halves, they continue constructing front halves. At the end of the day, six front halves are completed.

How many cars are there? Since $6 \times .5 + 1 = 4$, it seems as if ‘Four cars are completed’ should be true. However, it doesn’t seem true. Speakers differ in their judgements: some intuit that it is false, while others intuit that it is simply unacceptable/odd.

Reflection on the case yields the kernel of an explanation. Since the front-half and the back-half of a car play different roles in a functioning car, counting two front halves as amounting to a single car is unacceptable. This result can be reproduced for any structured object with functionally distinct components. Our theory of counting should both predict and explain it.

Q4: How and why is partiality-summation constrained?

2. Counting as a Type of Measuring

2.1 A Measure-Function Semantics for Number Words

Most familiar semantic proposals for number words deliver the same truth-conditions as our logic-class procedure. How can we give a semantics that accommodates partiality-sensitivity? Recent work on number word meaning gives us a clue. On the most familiar view, given in Barwise and Cooper (1981), number words are analyzed as quantificational determiners and they are taken to express relations between extensions. On this view, ‘three’ has the following meaning:

$$[[\text{‘three’}]] = \lambda P. |P \cap Q| = 3$$

For a variety of reasons, theorists have been steadily moving away from this approach. A approach defended by Krifka (1998) and Landman (2004), among others, takes number words to be predicates of pluralities. To a first approximation, ‘three’ on this view is true of three-membered pluralities.

$$[[\text{‘three’}]] = \lambda x. \#(x) = 3$$

An approach championed by Hackl (2000) takes number words to refer to numbers, but derives more complex meanings (whether predicative, quantificational, or whatever else) with a silent operator dubbed ‘many’. Here’s one proposal for ‘many’:

$$[[\text{‘many’}]] = \lambda n. \lambda x. \#(x) = n$$

On this approach, the tacit ‘many’ combines with the referential ‘three’ to produce the predicate meaning given above. Eventually, I will endorse a form of Hackl’s proposal, though I will give a slightly
Counting as a Type of Measuring

different meaning for the tacit ‘many’. In addition to the virtues cited by Hackl and followers, the proposal on which number words are referential allows us to give a straightforward account of anaphoric cases like (8) in which the anaphoric expression seems to refer to a number picked out by an earlier-occurring number word.

For the time being, though, set aside Hackl’s proposal and focus on the predicative meaning for ‘three’. Instead of taking the cardinality of a set, we have a function, picked out by ‘#’, that delivers the count of some objects (a plurality). If we take such a function to be involved in number word meaning, then there is an analogy between sentences that express counts and sentences that express measures — to a first approximation they both contain functions from objects to numbers/degrees.

To better understand this analogy, consider sentences like (14) that express measures.7

(14) Two liters of wine are in the bottle

The intuitive truth-conditions of (14) are clear: if we measure the volume of the wine in the bottle in liters, then the value is 2. This is different from individuating particular quantities of wine. So, we don’t want to incorporate any individuation into our semantics for (14); taking there to be a set of individuals that are each a liter of wine is the wrong way to go about things.8 Rather, we should take there to be a measure function involved: a function from some entity (or entities) to a value along a scale such that the relations between outputs of the function preserve actual relations between the objects being measured.

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7. More precisely, sentences like (14) can express measures. They can also be read individuatively. Rothstein (2009) and (2011) discusses these different readings in some detail. See, also, section 3 of Liebesman (2015).
8. One reason this is incorrect is that there are many more than two quantities of wine in the bottle that measure a liter. To see this, it suffices to realize that there are many different ways to divide a two-liter quantity of wine into sub-quantities that each measures a liter.
david liebesman

Counting as a Type of Measuring

(17) can be paraphrased as follows: there are some objects (a group/collection/plurality/whatever) such that they are two, they are birds, and they are singing, and any bird is one of them.

This initial proposal doesn’t cohere with our observation that the restrictor position of counting sentences is intensional.13 However, it can be slightly modified. Rather than ‘#' being a function from pluralities to numbers, we can take it to be a function from pairs of properties and pluralities to numbers. On this view, the semantic form of (16) is given by (18), where ‘B' designates the property of being birds.

(18) ∃xx(#<B,xx>=2 ∧ singing(xx) ∧ ∀y(#<B,y> > 0 → A(y,xx)))

We can paraphrase (18) as follows: there are some objects such that they are two birds, they are singing, and any bird is one of them.

More generally, the proposal is that a counting sentence of the form “N Fs are G’ has the form given in (19) on the exact reading, and (20) on the at least reading.

(19) ∃xx(#<Fx,xx>=n ∧ G(xx) ∧ ∀y(#<F,y> > 0 → A(y,xx)))

(20) ∃xx(#<Fx,xx>=n ∧ G(xx))

Given all of this, we’ll analyze number words (when combined with tacit ‘many’) as functions from pluralities and properties to values on a scale (the counting scale):

[[‘Two’]][[‘many’]] = λP.λxx.#(P,xx) = 2.14

The proposal has three clear departure points. First, it incorporates the # function on pluralities utilized in recent proposals about the semantics of number words, e.g. Landman (2004). Second, it takes that function to bear crucial analogies to functions utilized in the analyses of measure sentences, e.g. Rothstein (2009) and (2011). Third, it accommodates the data in 1.2 and follows Salmon (1997) in taking counting to be property-relative.15

The crucial component of this hypothesis about the meaning of number words is #, which takes a property and some objects as input, and outputs a number. One way to begin to understand # is by analogy with the phrase ‘unit of’. We can use ‘unit of’ to combine with mass nouns like ‘wine’ in order to count. Imagine that we are in a context in which the standard unit of wine is 750ml. If we have two full 750ml bottles and one half-full bottle in the fridge, there are two and a half units of wine in the fridge.

The analogy with ‘unit of’ gives us a start in understanding #, but in order to see how our theory of counting can answer Q₁ — Q₄ we need much more. The rest of the paper, in one way or another, attempts to answer a straightforward question: What is #? Since # is the crucial and mysterious semantic component in the semantic form of natural-language counting sentences, we can put our straightforward question (somewhat tendentiously) as follows: What is natural-language counting?

We can make a start by imposing some constraints on #. Here are two (I’ve omitted the second-order universal quantifiers binding ‘P’ and ‘Q’):

C₁: There are at least as many Ps as Qs iff ∃xx∀yy(#(P,xx) ≥ #(Q,yy))

C₂: If the sum of the numbers of some Ps (xx) and some other Ps (yy) is n, then #(P,xx) + #(P,yy) = n.16

13. To see this, note that intersubstitution of any predicate co-extensive with ‘birds’ into (17) would preserve truth-value.
14. On this proposal, number words are of type <<e,t>><e,t>>, which is the same type attributed to them by Ionin and Matushansky (2006), though for different reasons.
15. Salmon’s ‘preferred solution’ is the clearest influence, as it converges with both the first and third departure points. Salmon, however, does not go on to develop the solution in detail nor incorporate it into a semantic theory. (Nor were those his aims!)
16. Note that I’ve taken this to be a necessary condition on summation of Ps, and not a necessary and sufficient condition. There are two reasons for this: The
These are just two of the many restrictions one can reasonably make to ensure that the counting function # adequately reflects relations between the objects being counted. The latter restriction is a weakened version of a summativity constraint that familiar measure functions obey.17

Now we can give content to the titular claim that counting is a type of measuring. First, I’ve made a claim about the structure of counting sentences. Counting sentences like (16) have a similar semantic structure to measure sentences like (14). In particular, both contain a function from some objects to a value along a scale, and both express that that function yields a particular value for those objects. Second, I’ve made a claim about the function involved in counting sentences. I claim that it bears an analogy to the functions involved in measuring sentences. In particular, both sorts of functions are constrained by the fact that they must reflect actual relations between the objects they measure/count. This is just what Krifka (1990) meant by ‘measure function’. Given that, counting sentences contain measure functions just as measure sentences do.

2.2 Analyzing the Counting Function

C1 and C2 restrict the range of functions that are candidates for #. However, they fall far short of determining a unique function. Furthermore, they fail to give us any insight into the nature of #, insight that we should expect to help answer Q1 — Q4.

I will now turn to the more difficult task of analyzing #. I use the term ‘analyze’ strictly here, to mean separate into components, rather than in the loose sense of explicating. One component is a cardinality component of the sort that most usually take to exhaust the nature of counting. This component can only produce natural numbers. The second component is more complicated. Intuitively, for some property P, it assigns numbers to partial Ps. This component, for instance, assigns .5 to the half-bagel. To a first approximation, counts are provided by summing these components.

The approximation must be modified because the second component cannot produce numbers greater than or equal to 1. This restriction explains the puzzle case at the Toyota factory. Here, formally, is the proposal:

#(P, xx) = |\{xx ∈ P ↓\} | + the n < 1 such that ∑ \{xx \} ↓ f(P, y) is n

Let me now spell this out. # allows us to count some objects xx relative to a property P. # takes, as its input, some objects and a property, and outputs a sum of two values.18

The first value is relatively straightforward: it is the cardinality of the set of xx that are P at the relevant world. ↓ moves us from a property to the set of objects that instantiate that property at a given world. Assuming we’re discussing the actual world, ↓P is the set of actual Ps. For the sake of presentation, I’ve suppressed the world-sensitivity. Putting this all together, the first value is just the cardinality of the set of xx that are P, as desired.

The second value is more complex. It is itself a sum. Before specifying just what sum it is, notice that there is a restriction. The sum that provides the second value cannot be greater than or equal to 1.

I’ve captured this by specifying the sum with a definite description which will fail to designate if the value is greater than or equal to 1.

To generate the sum, we begin by specifying the set of objects yy we’re


18. The proposal I make has some key similarities with the proposal of Nicolas (ms.). In particular, both utilize measure functions, and both recognize the importance of standards/units. However, the details are quite different in a number of ways. Most prominently, Nicolas thinks this meaning is operative only in the case of fractions while I think it is always utilized in counting.
counting that are not Ps. We then use our function $f$ to map each individual $y$ and property $P$ to some number. The second value sums these numbers, as we'll see, is a partiality-measuring function.

Let's see this proposal in action. Consider our initial bagel scenario in which there are two whole bagels and a half-bagel on the table. We want to count the objects on the table, relative to the property of being a bagel. Our objects to be counted are bagel 1, bagel 2, and the half-bagel. Our property is that of being a bagel. The $f$ function takes these objects and that property as its input. Its output is a sum of two values. The first value is the cardinality of the set of objects that are actual bagels. This is \{bagel 1, bagel 2\}, so the first value is 2. The second value is the sum of $f(P, y)$ for each $y$ that is one of the objects, but not a bagel. There is only one such object: the half-bagel. Assuming partiality is determined by volume (roughly—a more detailed discussion is to come), $f(P, \text{half-bagel}) = .5$ because the half-bagel is .5 the volume of a standard bagel. Finally, we sum our first value — 2 — and our second value — .5 — to produce the total count of objects on the table relative to the property of being a bagel: 2.5. This is the desired result given that two and a half bagels are on the table.

Before moving on, it is worth responding to the worry that the proposal is offensively general. I've focussed on relatively few cases, such that in each of them it is obviously sensical to speak of partial Ks. However, as many have noted, partiality may seem strained when it comes to other kinds. Snyder and Shapiro (forthcoming), for instance, cite the oddity of the sentence ‘Fluffy had 6.34 kittens’ and remark ‘...it makes little sense to talk of fractional kittens, at least most of the time.’

According to the worry, only certain occurrences of number words are partiality-sensitive. However, when number words combine with terms like ‘cat’, they contain no partiality sensitivity. If we projected this worry into our semantics, the result would be that number words are ambiguous, sometimes expressing a partiality-sensitive meaning, other times expressing a counting-by-identity-style meaning.

There are two main reasons I reject this worry and the ambiguity view it suggests. The first is that an ambiguity view is unable to straightforwardly account for inferential connections between the purportedly different meanings. For instance, from (21) and (22), we can infer (23).

(21) Two bagels are on the table.
(22) Two kittens are on the table.
(23) The same number of kittens as bagels are on the table.

The second reason I reject the ambiguity view is that, as suggested by Snyder and Shapiro's 'at least most of the time', there are contexts in which partiality matters for counting just about any kind we come up with. Imagine that aliens have abducted several kittens, and, bowled over by their charms, the aliens want kittens of their own. However, they can't wait for nature to take its course. So, the aliens build a kitten-producing factory in which they embark upon a painstaking molecule-for-molecule construction of clones of their favorite. The factory-manager alien, it seems to me, can truly state that there are 6.34 kittens when the factory has completed construction of six, and is .34 of the way finished with construction of the seventh. (What is it to be .34 finished with the construction of a kitten? See section 3 for a discussion.)

This sort of factory-scenario is hardly limited to kittens. We can construct a nearly identical thought-experiment for any physical kind, as long as we imagine aliens with varying interests. This shows that few (if any) kinds are wholly immune from partiality-sensitivity.

2.3 Answering the Questions

Q1: What resources do we need to adequately capture the truth-conditions of counting sentences, and how do they go beyond first-order predicate logic with identity?

On the traditional translation procedure, (2) gives the truth-conditions for (1):

\[
\text{(2)} \quad \text{Two bagels are on the table.}
\]

\[
\text{Vol. 16, No. 12 (July, 2016)}
\]
(1) Two frogs are in the bog.

(2) \( \exists x \exists y (\text{frog}(x) \land \text{frog}(y) \land x \neq y \land \text{in the bog}(x) \land \text{in the bog}(y) \land \forall z ((\text{frog}(z) \land \text{in the bog}(z)) \rightarrow (z = x \lor z = y))) \)

If we analyze (1) as I’ve suggested, it is clear why (2) does not give its truth-conditions. On my view (1) has the truth-conditions given in (24), where ‘F’ designates the property of being a frog.

(24) \( \exists xx (\# F xx = 2 \land \text{in-the-bog}(xx) \land \forall y (\# F y > 0 \rightarrow \text{A}(y, xx))) \)

The # function is analyzed, in turn, as the sum of two components. The first is the cardinality of the set of xx that are frogs. The second is the sum of partial frogs. The universally quantified formula ensures that xx is the maximal set of things that are frogs or partial frogs.

Given these truth-conditions, it is clear why the translation procedure fails: it fails to account for the second component of the # function, which is sensitive to partial frogs. Furthermore, since we are taking counting sentences to have exact truth-conditions, the logic-class truth-conditions will always neglect this aspect of counting, whether or not there happen to be partial frogs. That said, when there aren’t partial frogs, the traditional truth-conditions will deliver the correct verdict. However, this will be due to luck: the truth-conditions will deliver the correct verdict only when partiality is irrelevant. I’ll return to this in discussing Q3.

In sum, the answer to Q1 is that we need both the intension expressed by the restrictor term, and the counting function. Once we have these resources we can provide adequate truth-conditions.

Q2: How can we explain the intensionality in the restrictor position of counting sentences?

In the analysis of #, the restrictor term makes two contributions. In calculating the cardinality of the set of xx that are F, it contributes a property that is then shifted to the set of actual things that instantiate that property. This role for ‘F’ yields no intensionality, since any extensionally equivalent properties will produce the same set. However, ‘F’ also plays a second role. f designates a function from properties and objects to numbers. Importantly, one of the arguments taken by f is the property expressed by ‘F’, not the set of things that instantiate that property. Given that co-extensional terms can express different properties, it follows that the # function is intensional in the sense that co-extensional terms provide it with different arguments.

To see how this works, recall our pair of sentences that motivated the intensionality of counting:

(3) Two and a half bagels are on the table

(9) Two and a half New York bagels are on the table.

In (3) the restrictor term is ‘bagel’, and in (9) the restrictor term is ‘New York Bagel’. Even if these terms happen to be co-extensional, they are intensionally inequivalent. This intensional inequivalence matters to measuring partial objects of bagel-kind and New York bagel-kind. If a half-bagel is from Montreal, it is a partial bagel but not a partial New York bagel. Partial New York bagels must be from New York. All of this shows that co-extensional predicates can make different contributions to measuring partiality. In our analysis, we’ve captured this by taking f to be a function from properties and objects, not a function from extensions and objects.

Q3: What accounts for the successes of the familiar translation procedure?

Earlier, I made a distinction between truth-conditions being correct and truth-conditions yielding the correct truth-value. I’ll now explain it. Yielding a correct truth-value is world-relative. A formula yields the correct truth-value for a sentence at a world w iff the sentence and the formula have the same truth-value at w. A formula gives the correct truth-conditions for a sentence iff, necessarily, it yields the correct truth-
value.\textsuperscript{19}

Our initial observation was that the logic-class truth-conditions seem to get things right most of the time. Now we can make this somewhat more precise. The logic-class truth-conditions yield the correct truth-value in lots of familiar circumstances. This is the observation that drives Q3.

Our analysis allows us to straightforwardly account for the successes of the traditional procedure. If we are counting some objects of a kind K, and none of those objects are partial Ks, then they will not contribute at all to the count. In that case, only the first component of the count — the cardinality of the set of counted objects that are K — will matter. Since calculating this cardinality is equivalent to counting only whole objects, the logic-class truth-conditions will yield the correct truth-value.

Q4: How and why is partiality-summation constrained?

It is easy to see that partiality-summation is constrained on my proposal. I’ve built into the # function that the sum of partial members of a kind cannot be greater than or equal to 1. If it is, the definite description will fail to designate. The result will be either falsity or lack of a truth-value, depending on one’s favored view of non-designating definites. Recall that in the car-factory case, intuitions were split as to whether ‘Four cars are on the floor’ was false or otherwise odd/unacceptable. This mirrors our reactions to non-designating definites.\textsuperscript{20} So, there is a sense in which Q4 is already answered. However, to make the proposal plausible, Q4 should be answered in a non-ad hoc manner.

To see that my answer to Q4 is not ad hoc, we can think more deeply about the components of the proposal. On the one hand, we have a mechanism for calculating the cardinality of a set. On the other hand, we have a mechanism for summing the values of various partial members of a kind. The total count is provided by summing the outputs of these independent mechanisms.

Ordinarily, the sum of two values doesn’t provide information about how that sum was reached. If I tell you that four animals are in the house, you don’t know whether it is three dogs and one cat, two of each, or some other distribution entirely. My proposal is that counting is different: a total count provides some information about how that count was reached.

To see this, consider the entailments of counts provided by mixed fractions. If two and a half cars are on the floor, it follows that a car is on the floor. More generally, a true count relative to K, provided by a mixed fraction, entails that there is a K.

If we were to let the two components of the # function interact, such that the sum of partiality-measures of Ks wasn’t limited to being less than 1, this entailment would not hold. Imagine that there are 16 front car-halves on the factory floor. It is false that there is a car on the factory floor. However, if we were to allow unrestricted summation of the measures of partial cars, we’d predict that it is true that eight cars are on the factory floor. This, in turn, would entail that a car is on the factory floor, which is false. In order to block the entailment, we restrict the summation of partiality-measures.

The upshot is that my answer to Q4 is not ad hoc, because it arises from an interaction constraint between the two components of our analysis of #. This interaction constraint, in turn, is manifested in entailment behavior.\textsuperscript{21}

Merely not being ad hoc hardly suffices for being true. I adopted a strong constraint on partiality-summation after considering just one example, so it is natural to worry that it falls prey to counterexample.

\textsuperscript{19}I’m not assuming that there’s nothing more to semantics than matching sentences with formulas that give the correct truth-conditions — quite to the contrary. However, the distinction is nonetheless useful.

\textsuperscript{20}See von Fintel (2004) for a clear presentation of the contrast and a theory on which falsity judgements are due to a fallback strategy rather than straightforwardly responsive to truth-value.

\textsuperscript{21}There’s been a huge amount of work on number concept acquisition. This work has separated a number of cognitive systems that play a role in our acquisition of number concepts. Theorists differ in just which system they take to
I’m not aware of any convincing counterexample, but there are three main types of *prima facie* counterexamples. I’ll now turn to these.

I place two whole bagels on the table. Later, in an attempt to be a good host, I slice each bagel in half and place the four halves back on the table. It seems that there are four partial bagels, summing to 2. We can provide a perfectly accurate count: two bagels are on the table. *Prima facie*, this count is ruled out by our restriction that the partiality-measure of half-bagels cannot be summed, if the sum is greater than or equal to 1.

Two important observations allow us to account for this example. The first is that a bagel doesn’t go out of existence when it is sliced in half, or even when the two halves are spatially separated. So, in the envisioned scenario ‘bagel’ has two entities in its extension, even if these are scattered entities. The second is that there must be a restriction on counting both an object and its parts at once. If there is no such restriction, then the theory that I have developed would be a disaster. Counting would always misfire because the parts of an object seem to sum to exactly 1! The form of this restriction on counting both a thing and its parts is controversial. I’ll return to it in section 5.

The first type of *prima facie* counterexample, then, fails because it ignores the fact that objects of the kind to be counted can be scattered. Furthermore, note that with enough background, we can count even a collection of unassembled parts as an object of a kind. For instance, in the Ikea warehouse, I can demonstrate a box of parts and truly utter ‘that’s a table’ (perhaps differentiating it from a chair). So, what often seems to be summation of partial members of a kind may simply be a case in which those objects actually constitute a single member. At the linguistic level, this amounts to nothing more than the familiar point that general terms are context-sensitive.

In section 1.4 I gave the kernel of an explanation for the Yaris case. I claimed that we cannot count two front-halves of a Yaris as a single car, because the front-half and back-half of a car play different functional roles in a complete car. Two front-halves together are functionally incomplete. We can connect that kernel to my responses to the first *prima facie* counterexample. One response was that objects may be scattered. Two front-halves, however, cannot be a scattered car. A second response was that unassembled parts may constitute a single kind-member in some contexts. However, even given a flexible context, they cannot constitute a member without being able to play all of the relevant functional roles.

Here’s another potential counterexample: I break my piggybank and change spills onto the table. I want to count the money, so I begin counting the quarters. Once I get to five, I can truly utter the following:

\[(25) \text{(At least)} \$1.25 \text{was in the piggybank.}\]

The apparent problem is that I have taken five items, each of which has a partiality measure of .25, and added them to reach a sum greater than 1. If that description is correct, it would show that there is no restriction that the sum of partiality measures be less than 1.

To properly understand this example, it is useful to consider Canadian currency. In Canada, single dollar coins are called ‘loonies’. (26) and (27) have different truth-conditions.

(26) Two loonies are on the table.

(27) Two dollars are on the table.

For (26) to be true, there must be two dollar-coins on the table. For (27) to be true, less is required: there must merely be currency that totals to two dollars. This contrast is linguistically important: (26)
supports plural anaphora. We can sensibly follow it with ‘they were each minted in 1996’. (27), however, does not support such anaphora. What the anaphora support makes clear is that when we utter a sentence like (26), we’re individuating the coins on the table. However, when we utter (27), we aren’t. Rather, we’re measuring, in dollars, the total amount of money on the table. This can be captured straightforwardly by taking (27) to have the same semantic form as a measure sentence like ‘Two liters of wine are in the bottle.’ Rather than employing the # function, both sentences employ other measure functions. In the case of (27), the measure function is the function from some objects to their amount-in-dollars according to the relevant monetary conventions. Since (27) doesn’t employ the # function at all, it can’t be a counterexample of my analysis to that function. All of this reasoning applies, mutatis mutandis, to (25).

Further evidence for this response is provided by the fact that there is a cross-linguistically robust complementarity between measure phrases and classifiers.22 If ‘$1.25’ is analyzed as measuring, we’d expect that it can’t combine with classifiers. Though the English classifier system is impoverished, we see support for this prediction. Compare (28) and (29).

(28) Two pounds of loonies are on the table.

(29) ?? Two pounds of dollars are on the table.

Insofar as we can make sense of (29), we must be reading ‘dollars’ individuatively, to be designating single-dollar coins or bills. Given the reading on which eight quarters are two dollars, (29) is nonsensical.23

The contrast between (26) and (27) shows that determining whether a sentence genuinely provides a count is not as simple as consulting our intuitions about that sentence. Rather, we have a battery of


23. Yet further evidence for the response comes from applying Rothstein’s (2009 and 2011) diagnostics for measure readings to (25).
D.A.R.E. COUNTING AS A TYPE OF MEASURING

Intonational stress on ‘two and a half’ and ‘two and a quarter and a quarter’. This is strong prima facie evidence that the negation in (30) is metalinguistic rather than truth-functional, which requires no special intonation.

In (30) it is natural to take the implicature that there are ‘two and a half’ cars in the normal way to be the object of the denial. By swapping ‘two and a half’ for the marked ‘two and a quarter and a quarter’, we block this implicature.

3. MEASURING PARTIALITY

Thus far I’ve said relatively little about how we give the partiality-measure of a particular object relative to a property, which is provided by the $f$ function in our analysis. In this section, I’ll dig deeper.

We can begin to better understand partiality measurement by reflecting on the variety of features that affect the counts provided by non-natural numbers.

• Spatial: One and a half bagels are on the table. (The bagel-portion is half the size of a normal bagel.)
• Temporal: Three and a half sessions are finished. (We’re taking a five-minute break before the Q&A session for the fourth talk.)
• Production: Two and a half cars are completed. (We’ve built two complete cars and have completed half the tasks on the checklist for production of the third.)

• Content: Two and a half of my firmest beliefs are about Sichuan food. (One of my firmest beliefs is that non-Sichuan food is never as good.)
• Function: Three and a third of the houses are habitable. (Three houses are completely habitable and a third of the rooms in the fourth are habitable.)

The diversity among the members of this list can make us skeptical about providing an analysis of measuring partiality. In fact, I do hold a skeptical view. To be clear, though I am skeptical that there is an analysis of partiality measurement, I also don’t think that such facts are fundamental or ungrounded. Rather, I merely think that we can’t give a pithy account — in other terms — of the function (the $f$ function) that calculates partiality.

While I doubt that an analysis is forthcoming, I do think there are a number of informative things we can say about partiality measurement. Reflection on the above list reveals two different requirements for something to have a partiality measure relative to a kind (e.g. to be a half-bagel or a half-car).

The first is that the entity must have a particular character. A piece of plastic that happens to be half the size of a normal bagel is not a half-bagel; a one-hour event in which nobody talks is not a half-conference session. Rather, in order to be a half-bagel, something must be a partial bagel to begin with, which, in turn, requires something like being made of yeast, flour, and water. The second requirement is that the entity must have a particular extent.

For an event to be a half-conference session, it must be an hour long — assuming we are at a conference with two-hour sessions. Notice that

25. As observed by Huang (2014), l-implicatures are not always subject to metalinguistic negation. However, some are. For instance: ‘John and Mary didn’t buy a BMW; John (long pause) and Mary bought a BMW’ cancels the l-implicature that they bought a BMW together (c.f. Huang 2014: 59). Almotahari (2015) makes a hypothesis about the availability of metalinguistic negation: it is available only if truth-functionally negating the denied clause is incompatible with the affirmation of the follow-up clause. On my view, this condition is met in (30).

26. Notice that (30) also satisfies Almotahari’s (2015) diagnostic for metalinguistic negation. His claim is that if the denial of the first clause is not incompatible with the denial of the second clause, then the negation is likely metalinguistic. This holds in the case at hand: there being two and a half cars is not incompatible with there being two and an eighth and an eighth.

27. cf. Yablo (2014) on content parts, though it is not clear how to extend Yablo’s machinery to account for half subject-matter.

28. Here I’m inspired by Szabó’s (2008) skepticism about giving a reductive account of what it is to be a thing-in-progress.

29. In this discussion, I’m freely moving between discussing properties and kinds. In fact, I think they are quite different; see Liebesman (2011). The differences, however, do not matter for the purposes at hand.
the most intuitive manner of understanding this is in terms of standards: we've been to half a session because we've been to an hour of this session, and sessions are standardly two hours long.

Utilizing these observations, we can begin to better understand the nature of partiality measurement. The character condition provides us with a lower bound: if an object $x$ fails to meet the character condition for a property $P$, then $f(x, P) = 0$. Property instantiation itself can provide us with an upper bound: if an object $x$ is $P$, then $f(x, P) = 1$. The extent condition will treat all of those cases in the middle: the cases in which the object is not $P$, but does meet the character condition. In 3.1 I’ll further investigate the character requirement, and in 3.2 I’ll discuss the extent requirement and, in particular, the manner in which standards are relevant. Again, I won’t give an analysis of either requirement or of partiality measurement, because I don’t think that such an analysis is possible. However, there are a number of substantial things we can say about both components, and these give content to the notion of partiality measurement utilized in our analysis in the form of the $f$ function. In 3.3, I’ll contrast the notion of partiality measurement used here with a proposal made by Ionin, Matushansky, and Ruys (2006). This will give me occasion to provide a compositional semantics for terms like ‘two and a half’ that respects both the complexity of these terms and the observations made thus far.30

3.1 Character

For something to measure .5 relative to kind Bagel, it must be a partial bagel to begin with. We can generalize this to capture the character requirement on partiality measurement as follows: $f(x, K) > 0$ iff $x$ is a partial $K$. This terminology exchange can help make investigation into the character requirement a bit more tractable. We can now ask what it is to be a partial $K$.

We want to understand the relation between an object and a kind $K$ that obtains when the object is a partial $K$; this relation is designated by the word ‘partial’ as we’ve been using it.31 Call this relation ‘partialhood’. We can gain some insight into partialhood by contrasting it with its more familiar cousin, parthood.

As already mentioned, there are clear extensional differences between parthood and partialhood. For an object to be a partial house, it is neither necessary nor sufficient that it is a part of a house. My water-heater is a part of my house but not a partial house; a halfway-built house is a partial house, but not part of a house. These extensional differences flow from some more fundamental differences between the relations.

Partialhood, unlike parthood, is primarily a relation between objects and kinds. In many familiar examples in which an object $o$ is a partial $K$, there is no member of $K$ to which $o$ relates. For instance, when we are a third of the way through building a house, there is no house present at all. Rather, we have a partial house. It is a partial house by virtue of its relation to the kind House, rather than any particular member of that kind. We can even imagine that the partial house is never completed, so it may never bear a relevant relationship to any member of house-kind. This stands in intuitive contrast with parthood. Intuitively, when some object $o$ is part of a house, there usually a house $h$ of which $o$ is part. (Whether this contrast bears scrutiny is not at issue here, I merely bring it up to help us better grasp partialhood.32)

Given that partialhood has different relata than parthood, we’d expect few of the properties of parthood to be shared by partialhood.

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30. One needn’t adopt my particular proposal to think that standards are relevant to the truth-conditions of some counting sentences. Nicolas (ms.) crucially utilizes standards (though he calls them ‘units’).

31. Interestingly, this same relation may be designated by ‘ing’ when it is used to express progressive aspect. See, e.g., Landman (1992) and Higginbotham (2004) for views on which the progressive expresses a relation between events and kinds of events.

32. In fact, I do think there are some cases in which it doesn’t bear scrutiny and ‘part of’ designates a relation that can’t be fundamentally object/object. Imagine that we’re at the car factory producing a new car — the Schmaris. It seems
Most of the familiar formal properties of parthood, e.g. reflexivity and transitivity, are not shared by partialhood. While partialhood is not transitive, it may share a nearby property. Take kinds to be hierarchically organized. So, for instance, the kind Oriole is a sub-kind of Bird. The following principle is prima facie plausible: if o is a partial K and K is a sub-kind of K′, then o is a partial K′. This does seem to hold in all of the cases we’ve been discussing. A partial house is also a partial building. I suspect that with some ingenuity, philosophers could generate counterexamples to this principle. However, the principle does give us some insight into partialhood: we should expect hierarchal relations among kinds to be relevant to the instantiation of partialhood even if strong universal generalizations like the one above are too broad to capture the relevance.

One could heroically attempt to analyze the partialhood relation in, e.g., modal or telic terms. A full refutation of such attempted analyses is out of place here, but it is worth expressing some skepticism. On the most natural modal analysis a partial K is something that would be a K if its creation were completed. It is easy to see that this analysis fails: a partial bagel may result from the destruction of a whole bagel, not a step of bagel-creation. One could then attempt to give a disjunctive modal analysis — something of the following sort: a partial K is something that would be a K if its creation were completed, or was part of a K. Not only does this analysis run into the problem of linking parthood and partialhood too closely, but it is refuted by cases in which we have partial Ks that are in no way linked to complete Ks. Consider, for instance, a factory that only produces bagel-halves.

Undermining some obvious modal analyses is one way to cast doubt on modal analyses, but there is a more general route. The route focuses on sentences in progressive aspect:

(32) Elise is building a house.

The first claim is that sentences in the progressive aspect express partialhood. The idea is that for a sentence like (32) to be true, there must be an event that is a partial house-building event. The second claim is that the progressive cannot be analyzed in modal terms. Here I draw on recent work by Kroll (forthcoming) in which he gives a detailed argument against the possibility of such analyses. These two claims entail that we cannot give a modal analysis of partialhood.

Obviously, both of the claims in this more general argument are controversial. I do not have anything to add to Kroll’s discussion, so I won’t consider the second claim here. The first claim is motivated by intuitive parallels between the meanings of the progressive and our notion of partialhood. Merely lifting a finger does not suffice to be building a house, even if lifting a finger is a part of a larger event of house-building. Rather, for an event to count as a partial house-building it must have the right sort of character. This character is not given by parthood any more than the nature of partialhood is given by parthood.

After rejecting a modal analysis of the progressive, Kroll advances a telic analysis. The idea is that (32) is true just in case there is an event that is teleologically directed at there being a finished house-building event. One could try to generalize this to give a telic analysis of partialhood. The idea would be that the telos of a partial bagel is a bagel; the telos of partial house is a house.

I also find this teleological analysis implausible. Return to our half-bagel factory that doesn’t aim to produce whole bagels. It seems strained to claim that any of the half-bagels produced at the factory are teleologically directed towards being whole bagels. Of course, one could give a positive account of telos on which this is the case, but absent such an account it is hard to see that it is plausible or motivated.
Given all of this, I’m skeptical of either a modal or teleological reduction of partialhood. Furthermore, the variety of kinds, and the variety of ways to be partial members of those kinds, makes me skeptical of any analysis. However, we can grasp the notion of partialhood both by considering examples and by contrasting partialhood with the more familiar notion of parthood. Since the character condition on partiality measurement is understood in terms of partialhood, all of this gives us insight into the character condition.

### 3.2 Extent

For something to be a half-bagel, or a half-conference session, it must not only be a partial bagel or conference session; it also must have a certain extent. In the case of the bagel, roughly, it must be half the volume of a standard bagel. In the case of the conference session, roughly, it must be half the duration of a standard session.

One of the reasons that these initial glosses are rough is that there are two ways to satisfy the extent condition. On the one hand, something could be a half-bagel because it is half of a future or former bagel. In this case, it is straightforward to see how some x could be such that \( f(x, B) = 0.5 \) (where B is Bagel-kind): x is a partial bagel — so it satisfies the character condition — and x is the remainder of bisecting a whole bagel, or what would remain if a future bagel were bisected. This can easily be generalized to temporal extent, for the case of sessions. It could also be generalized to other measures of extent, e.g. production processes. In any of these cases, partiality measurement is fairly straightforward due to the fact that there is some actual object against which the measurement can be made. All that remains is to determine the sort of extent at issue. However, that will be determined by the kind in question.

The more complicated way something can be a half-bagel involves the alluded-to notion of a standard. Return to our half-bagel factory that doesn’t aim to produce whole bagels at all. None of the half bagels it produces will be parts of whole bagels. However, each of them is such that \( f(x, B) = 0.5 \). What accounts for this? The outline of an account is straightforward: each of them is half the volume of a standard bagel.

All of this suggests an account: take standardization to provide a function from kinds to sets of properties. Among these properties are extent standards: e.g. dimensions for the standard bagel. Partiality measurement could then simply be sensitive to these standards.

This account is intuitive, but it is too simple. As brunch aficionados know, Montreal bagels are smaller (and sweeter) than New York bagels. Given that there are multiple different kinds of bagels, we have no good reason to think that there is a single set of dimensions that provides bagel standards. In order to account for this, we must think of standardization differently. Rather than taking standards to be given by functions from kinds to sets of properties, we can think of standards as functions from kind/object pairs to sets of properties. The object is used to determine a sub-kind. So, for instance, the pair \((x, B)\), where x is a New York bagel, will take you to one set of standards, whereas the pair \((y, B)\), where y is a Montreal bagel, will take you to another.

Investigating just how standards are determined by kind/object pairs is an immensely difficult task. Again, I doubt there is a fully general analysis. Rather, we can grasp the notion by relying on key examples. At Philosophy conferences a standard session is two hours long. At English conferences, a standard session is 90 minutes long.

There are worries. Perhaps, in some cases, there is a lack of standards. Or, perhaps, standards aren’t precise enough to determine a measure function. I’ll return to these worries in 4.2 and 4.3.

### 3.3 Comparisons

Ionin, Matushansky, and Ruys (2006) give an explicit semantics of fractions, as well as a compositional semantics for terms like ‘two and a half’, that is continuous with the compositional semantics for complex cardinals like ‘twenty-two’ found in Ionin and Matushansky (2006).

Here is their proposal for ‘half’:

\[
\text{[‘half’]} = \lambda P. \lambda x. \exists y (P(y) \land \forall z (P(z) \rightarrow z \leq y) \land \exists S (\Pi (S)(y) \land x \in S)
\]
\( \exists \mu (\mu \in M \land \forall s_1, s_2 ((s_1, s_2 \in S) \rightarrow (\mu(s_1) = \mu(s_2)))) \), where \( M \) is a contextually determined set of measure functions.

Their proposal employs the notion of a partition. \( \Pi(S)(y) \) is read as "set \( S \) is a partition of \( y \)." A partition, intuitively, is a way of dividing an object into sets of non-overlapping parts that jointly capture all of the object (cover it).\(^{33}\)

On their proposed semantics for 'half', something will be in the extension of 'half-bagel' just in case there is a three-membered set, such that the sum of its members is a bagel, and each of its members has the same measure, according to one of the contextually determined measure functions.

This semantics differs dramatically from the one I’ve proposed. On my proposal, an object could be a half bagel even if there is no bagel of which it is part. This is a positive result, given that the fact that there are two and a half bagels on the table does not entail that three bagels exist. This, then, is a problem with their proposal.

More problems are revealed when we consider their semantics for 'two and a half bagels'. As already stated, their proposal for 'two and a half' is continuous with their proposal for complex cardinals like 'twenty-two'. Both are analyzed as containing conjunction. In the latter case, 'twenty-two books are on the table' is analyzed as expressing 'twenty books and two books are on the table'. 'Two and a half' bagels, in turn, is taken to express 'two bagels and a half bagel', where 'half bagel' is analyzed as above.

I already, in effect, objected to this view early in my discussion. Recall that this proposal fails to do justice both to the existence of anaphora as in (8) and to the truth-conditions of (3).

(8) Two and a half bagels are on the table. Twice as many onions as that are on the counter.

The problem that Ionin et al. have with (8) is that the value 2.5 seems to be available for anaphoric reference, but, on their view it is not designated by any constituent of the sentence.

(3) Two and a half bagels are on the table

Another problem for Ionin et al. is that they fail to rule out the circumstance in which the half-bagel overlaps the two bagels. Imagine that you slice two bagels in half and put all four halves on the table. Ionin et al. predict that (3) is true. They are aware of this oddity and attempt to account for our intuition that (3) is false in such a scenario by appealing to its pragmatic oddity, in particular the violation of Grice’s maxim of manner. However, as they are aware, such an explanation would predict that we can override the oddity and achieve the reading. Unfortunately for their analysis, this does not seem to be the case with (3).

A virtue of the Ionin et al. analysis is that it gives an independently motivated compositional derivation of the meanings of complex cardinals and mixed fractions. Their idea is that there is an independently available type of conjunction that allows us to predict that 'two and a half P' can express 'two P and a half P'. Abandoning their analysis forces us to explain how we can recover its virtue of being fully compositional and utilizing only independently motivated mechanisms.

Recall that, on the view developed here, to a first approximation, 'two and a half' has the following semantics:

\[
[[\text{Two and a half}]] = \lambda P \lambda xx (#(xx, P) = 2.5)
\]

Our aim is to derive this compositionally, in an independently plausible manner.

To do this, begin by observing that 'and' in English has a use on which it expresses summation, as evidenced by the fact that (33) can be glossed in English as (34).

(33) 2+2=4
(34) Two and two are (is) four.

---

33. More carefully, \( \Pi(S)(x) \) iff \( S \) is a cover of \( x \) and none of the members of \( S \) overlap. \( S \) is a cover of \( x \) iff \( S \) is a set of subparts (proper parts) of \( x \) and every subpart of \( x \) is a subpart of a member of \( S \).
We also see this in the case of mixed fractions, as in (35) and (36).

(35) $3 + \frac{1}{4} = 3.25$

(36) Three plus a quarter equals three and a quarter.

These examples motivate the existence in English of a sense of ‘and’ that expresses summation. It can be straightforwardly captured as follows:

\[
\text{\textquotesingle and\textquotesingle} = \lambda x \lambda y. x + y
\]

Throughout this discussion, I’ve been primarily working with a semantics of number words on which they are type $\langle e, t \rangle \rightarrow \langle e, t \rangle$. However, as mentioned earlier, there is a view championed first by Hackl (2000) and subsequently by theorists such as Kennedy (2013) on which number words refer to numbers and they combine with a tacit $many$ operator to derive higher-order meanings.

With these resources, we can give a compositional semantics for ‘two and a half’. On the view, ‘two’ refers to 2, ‘half’ refers to $\frac{1}{2}$, and conjunction has the summation semantics given above. This then produces a term that refers to $2.5$. Subsequently this combines with the tacit $many$ operator, which has the meaning given below, to derive our proposed value.

34. I am assuming that ‘three and a quarter’ in (36) and ‘two and two’ in (34) are referential. This is controversial, as discussed by both Hodes (1984) and Hofweber (2005). If, following Hofweber (2005), one takes these terms to be quantificational, rather than referential, then I would have to give an alternative analysis of the summation use of ‘and’. This is fairly straightforward to do, here’s one way to go. Define a function V, which takes generalized quantifiers that express cardinality as input and outputs the cardinality they express. So, V(two) = 2. Then we can define the summation use of conjunction for quantificational meanings as follows:

\[
\text{\textquotesingle and\textquotesingle} = \lambda f. \lambda f'. \lambda P. \lambda x. \#(x, P) = V(f) + V(f')
\]

That said, I prefer the view Hackl-inspired view given here because it easily makes sense of the anaphora data, whereas a view on which ‘two and a half’ is non-referential doesn’t.

35. We may wish to extend the considerations in this paper beyond number words to other quantifiers. Armed with $\#$ and $f$, this is usually straightforward. For instance, if we want a sense of ‘all’ on which ‘all bagels are on the table’ is false if a half-bagel (which isn’t a bagel) is on the counter, then we can provide the following meaning for ‘all’.

\[
\text{\textquotesingle all\textquotesingle} = \lambda P. \lambda Q. \{x: f(P, x) > 0\} \subseteq \{x: f(Q, x) > 0\}
\]

I am not endorsing this view here, as it would take substantial additional investigation to evaluate its merits, but it is worth noting the possibility of extending the results from this discussion.

36. The thought that there’s tacit conjunction in ‘twenty-two’ is bolstered not only by the cross-linguistic data that Ionin and Matushansky cite, but also by the fact that in Victorian English, it was common to use ‘two and twenty’ instead of ‘twenty-two’.

\[
\text{\textquotesingle many\textquotesingle} = \lambda n. \lambda P. \lambda x. \#(x, P) = n
\]

This proposal is fully compositional, and is not susceptible to any of the objections levelled against the Ionin et al. proposal. Furthermore, we can extend the proposal to account for complex cardinals by following Ionin et al. in taking there to be tacit conjunction — provided by asyndetic coordination — present in the case of complex cardinals. (See Ionin and Matushansky 2006: section 4.)

In addition to allowing a compositional account of mixed fractions that is continuous with complex cardinals, there is another, more fundamental similarity between the account advanced here and that advanced in Ionin et al. (2006). On both accounts, counting to two and a half is complex. I’ve captured this by taking the counting function to be analyzed into two distinct components.

It is reasonable to worry that the account is needlessly complex. If there is a simpler account that gives satisfying answers to Q1 — Q4, perhaps we should prefer it. On an alternative proposal, $\#$ is taken to be a simple unanalyzed measure function.

In this case, simplicity is not a virtue. There are two reasons that taking $\#$ to be analyzed into two distinct components is explanatorily powerful.
The first reason is that our analysis of # helps us to understand partiality-summation is constrained. On the view advanced, the constraint arises as an interaction effect between the two components of #. In particular, counting requires that these components contribute in different ways to overall counts: the former providing cardinal numbers, the latter only providing non-cardinals.

The second reason is that our analysis of # helps us to better understand the intensionality of counting. I argued for intensionality by providing co-extensional kind terms that could not be intersubstituted in the restrictor position of counting sentences salva veritate. The reason for the failure of intersubstitution was that, while the kinds were co-extensional, they had different partial members. In particular, the bagels can be the same as the New York bagels, even though there is an object that’s a half-bagel but not a half-New York bagel.

Given that counting exhibits this sort of intensionality, a natural question arises: Does counting exhibit other sorts as well? In other words, are there examples of intensionality in the restrictor position of counting sentences that do not involve partiality? Strikingly, it appears that the answer to this question is no.

Ideally, our analysis not only makes counting intensional, but it also generates the precise sort of intensionality involved in counting. So, the ideal theory limits the intensionality of counting to cases involving partiality. This is exactly what our analysis does. Counting involves two separate components, and only the partiality-measuring component involves intensionality. We then correctly predict that intensionality will arise only when partiality is involved. A less complex analysis of # does not have the resources to identify the precise source of intensionality in order to yield this prediction.

4. Overlap, Imprecision, General Standards, and Lewis’ Examples

In analyzing the # function, I’ve neglected some familiar issues that have puzzled philosophers and linguists concerned with counting. I’ll conclude by discussing four such issues, and explain how my proposal handles them.

4.1 Overlap

The first issue concerns overlap. Consider a case in which there is one completed car on the factory floor, and nothing else on the floor. (37) should be true.

(37) One car is on the floor.

However, there’s a risk that, on the approach I’ve developed, (37) either is false or lacks a truth-value altogether. Consider the upper $2/3$ of the completed car, as well as the lower $1/2$. One could reason as follows: Each of these is a partial car. Relative to a spatial choice for $f$, the former is $2/3$ of a car and the latter is $1/2$. When we add these together, we get a value greater than or equal to 1. Therefore, on the view developed, (37) has no truth-value. Or, similarly, one could simply focus on the upper $2/3$ and reason that (37) is false because the set of cars has cardinality one, and there is some partial car.

The problem is clear: we are counting both the completed car and its parts. Given the multiple ways in which a car can be decomposed into parts, those parts — if they are partial cars in the sense relevant to the $f$ function — will sum to greater than 1. The solution is also clear: we need some way to prevent counting both the car as well as its parts.

Luckily, the need for some sort of mechanism to prevent the double-counting of an entity and its parts seems general, and not tied to my particular theory. For instance, here is Kratzer (2012: 168):

A fundamental principle of counting says that a domain for counting cannot contain non-identical overlapping individuals.

There are several different ways to develop such a mechanism. I’ll now survey three, the final of which is my preferred option.

The first way to prevent double-counting a thing and its parts is already suggested in the Kratzer passage: to require that the objects being counted do not overlap. This would obviously solve the problem in the case at hand. Since the parts of the car overlap the car, they could not be among the objects counted. (37) would, then, be true.
There are problems with such an approach. It seems that all sorts of familiar objects can overlap and, despite that, be counted perfectly well. Two townhouses can share a furnace. Requiring non-overlap among the counted objects is just too forceful.

The second way to prevent double-counting is more specific to our approach. Recall that the # function contains two components. The first component takes the cardinality of the set of actual Ps (for the relevant property P). The second component sums the f-values of the non-Ps, relative to P. We could prevent double-counting a thing and its parts by slightly modifying the second component. Instead of the second component calculating the f-value of non-Ps, it could calculate the f-value of non-P-parts (including improper parts). This wouldn’t run into trouble when it came to overlapping townhouses, but it would still deliver the correct verdict for (37).

However, one may reasonably feel that this approach, which builds a non-overlap condition into the semantics, is still too forceful. If there are cases in which partial Ks are parts of Ks, and we count both, then this approach fails. I don’t know of any uncontroversial case in which this occurs, but here’s a controversial one from Lewis (1993):

You draw two diagonals in a square; you ask me how many triangles; I say there are four; you deride me for ignoring the four large triangles and counting only the small ones. But the joke is on you. For I was within my rights as a speaker of ordinary language, and you couldn’t see it because you insisted on counting by strict identity. I meant that, for some w, x, y, z, (1) w, x, y, and z are triangles; (2) w and x are distinct, and ... and so are y and z (six clauses); (3) for any triangle t, either t and w are not distinct, or ... or t and z are not distinct (four clauses). And by ‘distinct’ I meant non-overlap rather than non-identity, so what I said was true (179).

Lewis’ moral from this example is that we sometimes count by non-overlap, not by non-identity. I’ll return to this shortly. However, for now, focus on a different upshot of the triangle example. Insofar as there is a context in which ‘eight’ is the correct answer to the ‘how many triangles’ question, it seems as if we are free in some contexts to count entities in addition to their parts. The second way rules this out.

The third way is the most subtle. It requires invoking our distinction between partial Ps and parts of Ps. There are lots of parts of my house: a basement, kitchen, bottom-half, etc. However, these parts of my house are not partial houses. Paradigmatically, partial houses are created in the process of building complete houses. Once houses are completed, then (fortunately) the partial houses no longer exist. Given this, we can prevent double-counting a thing and its parts without directly building a constraint into the semantics itself. Rather, the idea is that f(C,th)=0, where C is the property of being a car, and th is the top half of the car on the floor. This follows from two claims that I’ve already defended: (1) if f(C,th)>0, then th is a partial C (car), and (2) the top half of a car is not a partial car (though it is a part of a car). If this is correct, then we have no problem accounting for the truth of (37). As I said, my preferred way is the third. However, fully developing it requires saying more about what it is to be a partial member of a kind. So, I adopt it only tentatively here. The more general point is that most agree that there must be some mechanism that prevents us from overcounting a thing and its parts. Whatever the mechanism is, it could be applied to the case at hand.

4.2 Imprecision

The second issue concerns imprecision. Slicing a bagel in half is pretty easy. All success requires is that you cut it somewhere down the middle. There’s a worry that, on my proposal, slicing a bagel in half is exceptionally difficult. Imagine that the standard-sized bagel weighs exactly 2.98 oz. Assuming that we’re in a context in which we’ve invoked the spatial f function, it looks as if we’d need to produce an extremely accurate scale in order to slice a bagel in half. After all, we’d

37. Being a partial K, for some kind K, is a maximal property in the sense of Sider (2001).
need to produce a partial bagel that weighs exactly 1.49 oz. It gets even worse: unless our initial bagel was exactly 2.98 oz, at most one of the partial bagels can be a half bagel! After all, the other won’t be exactly 1.49 oz.

Before addressing this worry, note that a closely analogous case has been discussed in detail by Lasersohn (1999):

Suppose, for example, that I tell John that Mary arrived at three o’clock. In certain relatively unusual circumstances, the exact second of her arrival might be important, but most of the time this level of precision is not required. So if John finds out later that Mary didn’t arrive at three but at fifteen seconds after three, it would be unreasonable of him to complain “You said she came at three!” (522)

Lasersohn’s view is that, in uttering ‘Many arrived at three o’clock’, he said something false. Mary didn’t arrive at three: she arrived fifteen seconds after three. However, Lasersohn notes that in usual circumstances it is odd to speak with such precision or to correct somebody who doesn’t. So, while the utterance expresses something false, it is perfectly appropriate.

To explain this appropriateness, Lasersohn takes context to associate each expression with a set of values that are close enough to its semantic value — this is the expression’s pragmatic halo. Given these halos, each utterance will be pragmatically associated with many propositions beyond what it semantically expresses. As long as one of these is true, the utterance expresses something true enough for the purposes of the discourse. In the example, if the halo contains the proposition that Mary arrived at fifteen seconds after three, then the utterance is appropriate.

We can utilize Lasersohn’s machinery to make sense of the difficulty of slicing bagels in half. In most ordinary contexts, our interests don’t require perfect precision in bagel-slicing. Rather, anything within a range is good enough. ‘Half’, then, while denoting .5, will have a pragmatic halo that extends in both directions. As long as the size of the relevant partial bagel is within this halo, it will be perfectly appropriate to call it half.

4.3 Standards

The third issue concerns counting criteria. The previous subsection addressed the worry that standards may be overly precise. The worry here is that, for some properties, there may be no standards at all. The obvious cases of missing standards concern very general terms like ‘thing’. It seems clear that there is no standard thing. However, there are contexts in which sentences like (38) express truths:

(38) I did two and a half things before lunch and I’m hoping for one and a half this afternoon.

Imagine that my to-do list consists of the following: washing my dog, grading two student papers, and e-mailing my colleague. If, in the morning, I wash my dog, grade a student paper, and grade half of the next student paper, it seems that (38) is true. How can we account for this?

Context-sensitivity is relevant here. ‘Thing’ is clearly context-sensitive. In this case it seems to be true of things on my to-do list. What this shows is that standards will likely be determined not simply to the property expressed by the restrictor term, but a more specific property expressed by the restrictor term along with its contextual restriction. Exactly how this happens is a matter of some controversy, though Stanley and Szabó (2000) give a proposal.38

In some philosophical contexts, ‘thing’ can be used in an utterly unrestricted manner. How can we account for these cases? They seem especially challenging given the implausibility of taking there to be a

38. Stanley and Szabó focus on the extensional, giving a model of how the extensions of general terms are restricted in context. However, they rightly note (2000: 252) that in order for their proposal to generate an adequate modal profile for sentences containing domain restriction, there must be a way for the property expressed by general terms to somehow be restricted to a more specific property.
standard extent that covers every thing! The proper response to this worry requires appreciating that standards that determine partiality aren’t merely relativized to kinds: they are relativized to kinds and objects. This explains how we can account for the variation in standards between New York bagels and Montreal bagels. There are (at least) two ways to be a standard bagel and each determines a different extent. To understand how there can be standards associated with ‘thing’, begin by appreciating that there are as many ways to be a standard thing as there are things. In our framework, we can simply associate each pair of an object $x$, and $\text{Thing-kind}$, with the extent of $x$. This ensures that everything is a standard thing. In other words, the standards associated with ‘thing’ are utterly vacuous, as desired. However, we can distinguish ‘thing’, which imposes vacuous standards, from those words that don’t impose any at all.

4.4 Almost One
Those familiar with the philosophy literature on counting are likely surprised that I haven’t discussed Lewis’ case that we don’t count by identity. Lewis has us consider a number of examples in which, it seems, counting sentences are acceptable despite failing to satisfy their counting-by-identity truth-conditions. Here’s one:

If an infirm man wishes to know how many roads he must cross to reach his destination, I will count by identity-along-his-path rather than by identity. By crossing the Chester A. Arthur Parkway and Route 137 at the brief stretch where they have merged, he can cross both by crossing only one road. (1976: 27)

For reasons spelled out in Liebesman (2015), this case does not uncontroversially exemplify what Lewis has in mind. It is clear, however, just what he has in mind. There are some cases in which we seem to disregard massively similar objects of a kind $K$. Exactly what this similarity consists in is more controversial. Lewis focuses on mereological overlap, though Sutton (2015) makes a strong case that we should focus on functional overlap. The idea is that when counting some objects of a kind, if two or more objects bear some similarity relation to one another, then we count those objects as one. Lewis thinks this is what goes on in many instances of the problem of the many, though that’s more controversial.

Fitting such cases into the framework developed here is straightforward. Take the relevant relation that groups some objects as one (be it massive overlap or functional overlap) to induce an equivalence relation on the $K$s. We can then use this equivalence relation to partition the $K$s. So, if we’re counting clouds, we can sort the many clouds into relatively few equivalence classes. Now, instead of taking the first component of the # relation to provide the cardinality of the set of clouds, take it to provide the cardinality of the set of sets of equivalence classes. Voila: we’ve now produced Lewis’ more discriminating count. We can take these more discriminating counts to be generally available via a mechanism that shifts the meaning of # to the more restricted meaning that depends on a partitioning function that is contextually available. Right now I tend to think — with Lewis — that this process is pragmatic, though it would also be straightforward to make such a process semantically available.

40. One worry is that there are Lewis-type cases in which we seem to count by a relation more fine-grained than identity. For instance, if there is a street in the shape of a horseshoe and I have to cross it twice on my way to work, I may truly utter ‘I must cross two streets on my way to work’. The machinery I develop here cannot account for these examples, though other machinery I discuss in Liebesman (2015) can.

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