The 3D Tetrahedral Digital Waveguide Mesh with Musical Applications

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Abstract

The 2D rectilinear digital waveguide mesh algorithm to simulate wave propagation in the ideal memran was introduced three years ago as a multiply-free, parallel computation scheme suitable for high speed hardware implementation. Since that time, various alternative structures and add-on elements have been developed to make the mesh musically useful. We review some of these developments and outline the new tetrahedral mesh structure which now permits efficient multiply-free simulation of wave propagation in 3D space.

1 Background

The fundamental intuition-building observation to make, in order to understand how the waveguide mesh algorithm works, is that when you kick a chicken wire fence, waves seem to propagate on it much as on an ideal membrane. The chicken wire fence, like the waveguide mesh, is a regular interconnection of short vibrating string elements joined at nearly lossless scattering junctions. In [9, 10] we showed that a rectilinear arrangement of 4-port junctions is mathematically equivalent to the standard finite difference equation approximation to the lossless 2D wave equation. In fact, there are a variety of regular geometric mesh structures which compute valid difference approximations to the wave equation, for example, the hexagonal 3-port structure shown in Figure 1.

The 2D digital waveguide mesh has proven to be effective in the modeling of musical membranes and plates, particularly in combination with recent simplifications in modeling stiffness [8], non-linearities [7], and felt mallet excitations [7]. One of the more interesting musical applications is the waveguide mesh gong model, a section of which is illustrated in Figure 2: The J's mark ordinary 4-port lossless scattering junctions, in which the four inputs are summed and scaled by one half to form the so-called junction velocity, and the four outputs are computed by subtracting the respective inputs from this junction velocity. The PNF's mark passive nonlinear filters attached at the upper right. The passive nonlinear filter is, in its simplest form, an first order first-order filter whose coefficients are varied between two values depending on the sign of the filter state value [7]. This fi-
ter structure was developed in collaboration with John R. Pierce, to model the passive, nearly lossless, spreading of energy between modes of vibration in certain important classes of musical instruments. These all-pass structures may also be used to simulate a "stretching" of the modal frequencies due to stiffness [8]. In the center of Figure 2, 3 marks a special time varying scattering junction which is attached to a wave digital hammer mallet model through the signals marked $v^n_0$ and $v^n_i$. The wave digital hammer simulates the nonlinear compression forces and hysteresis in the soft mallet or piano hammer [7].

![Figure 2: The Waveguide Gong](image-url)

Davide Rocchesso and Federico Fontana have proposed a new percussion instrument based on an efficient 8-port triangular mesh structure with a wave digital air-loading filter and wave digital mallet port at each junction [1].

The 3D 6-port rectilinear extension to the mesh had been hypothesized [6], and was applied to the study of room acoustics by Savioja, Rinne, and Takala [2]. Tim Stilson first implemented the 3D rectilinear mesh to study wave propagation in a bent tube [4]. Figure 3 shows several frames from Stilson's animation of a pressure pulse plane wave in an acoustic tube trying to make its way around a U-turn. The computation was actually performed using a dense 3D mesh, and then the results were consolidated into a 2D image representation. Notice that as the pulse pressure wave develops as some of the pulse reflects and inverts off the turning wall of the tube. This is followed by some significant coupling of energy into the cross-sectional modes of the tube as the pulse continues around out of the turn.

The simulations by both Stilson and Savioja's team used a rectilinear 3D mesh computational structure. However, such a structure requires the

![Figure 3: Stilson's Animation of a Pressure Wave in a Tube Using a Dense 3D Rectilinear Mesh](image-url)
use of 6-port scattering junctions, which make a multiply-free implementation impossible in the isotropic case. Implementation of multiplies in high speed hardware can get expensive, and reduces the practicality of a densely sampled parallel mesh implementation useful for room acoustics or accurate physical simulations. The 4-port scattering junctions of the 2D mesh requires only an internal divide by 2, which could be implemented as an inexpensive right shift in binary arithmetic. However, the 6-port junction requires a divide by 3. The multiply-free cases occur for N-port junctions in which N is a power of two [3].

We describe here a tetrahedral distribution of multiply-free 4-port scattering junctions filling 3-space much like the molecular structure of the diamond crystal, were the placement of the scattering junctions corresponds to the placement of the carbon nuclei, and the bi-directional delay units correspond to the four tetrahedrally spaced single bonds between each pair of nuclei. Figure 4 illustrates the structure. The tetrahedral mesh is mathematically equivalent to a finite difference approximation to the 3D lossless wave equation. The frequency- and direction-dependent plane wave propagation speed dispersion error is comparable with that of the rectilinear mesh structure; however, computational and memory requirements are much improved in the tetrahedral structure, and now within the realm of practical high speed hardware implementation. The authors are grateful to Prof. Wen-Yu Kuo of Chung-Hua Polytechnic Institute of Taiwan for fruitful discussions on the structure and implementation of the tetrahedral mesh [11].

2 What is Dispersion Error?

The term dispersion is somewhat overloaded, but in the field of finite difference approximations it refers to an error in the speed of travel of waves. For example, in the solution to ideal membrane equation, waves of all frequencies travel at the same speed in all directions. However, in the standard difference approximation, and, therefore, in the equivalent rectilinear 2D waveguide mesh, plane waves travel at slightly different speeds depending on their frequency and on their direction of travel relative to the orientation of the mesh. In fact, it was shown [10] that all waves traveling in the diagonal directions travel at the same correct speed, but that waves traveling in the directions of the grid axes travel a little slower at higher frequencies. Figure 5 shows a circular wavefront expanding on a 4-port rectilinear mesh. Observe how the wavefront has remained sharp along the diagonal directions, whereas it has smoothed out along the axes directions, resulting from the higher spatial frequencies lagging behind.

Figure 6 and 7 illustrate a frequency domain view of wave speed dispersion in various two-dimensional mesh structures. These plots were calculated using a method which will be described in more detail for the 3D tetrahedral mesh case in the succeeding sections. The upper right plot in Figure 6 shows contours of the normalized wave travel speed on the 4-port mesh versus plane wave frequency and direction. The center region of the plot corresponds to low plane wave frequencies; the outer regions of the plot correspond to higher plane wave frequencies. The angular position on the plot, as seen from the frequency plane origin (at the center), corresponds to the direction of plane wave travel on the mesh. Notice that in the diagonal directions of the 4-port mesh, all frequencies travel at full speed, whereas the contour lines show that wave travel speed falls off along the axes directions as frequency increases, i.e., nearer the outer edges of the plot. The contour lines are marked off in 1% intervals at 90%, 95%, etc., of full speed. (The dark circles indicate the maximum useful plane wave frequency and will be explained in a subsequent section.) In the lower right of Figure 6 a contour plot of wave speed dispersion in the multiply-free 8-port rectilinear mesh structure is compared. In the 8-port case, things improved over the 4-port case in the axes directions, but got much worse in the diagonal directions.

Figure 5: Time Domain View of the Effects of Dispersion Error in the 4-port Rectilinear Mesh
Figure 6: Comparison of Dispersion Error in the Multiply-free 4-port and 8-port Meshes

Figure 7: Comparison of Dispersion Error in the 3-port Hexagonal and the 6-port Triangular Meshes
from the direction of \( \vec{r} \) is equal to the output from \( \vec{r} \) delayed by one sample. In the \( \mathcal{Z} \)-transform do-
main we write this relationships as,

\[
V_k^{z^2} = z^{-1} V_k^{z^2}.
\]

Using (2) and (3), we obtain an expression for the input signal to junction \( A \) from the \( \vec{r} \) direction in terms of the junction velocities \( A \) and \( \vec{r} \) only,

\[
V_k^{z^2} = z^{-1} V_k^{z^2} = z^{-1} (V_k - V_k^{z^2})
\]

\[
= z^{-1} [V_k - z^{-1} (V_k - V_k^{z^2})]
\]

which implies,

\[
V_k^{z^2} = \left( \frac{z^{-1}}{1 - z^{-2}} \right) (V_k - z^{-1} V_k)
\]

We substitute (6) into (1) to get an expression for the junction velocity \( V_k \) in terms of the four surrounding junction velocities \( V_l \).

\[
V_k = \frac{1}{2} \left( \frac{z^{-1}}{1 + z^{-2}} \right) \sum_l V_l
\]

Unfortunately, the orientations of the tetrahedra vary from point to point. In Figure 9 the tetra-
he-dron around point \( A \) and that around point \( B \) are in vertically opposite orientations. However, con-

Consider the relationship between the center point \( A \) and the twelve equally spaced junctions marked 1 through 12, which are all equidistant from \( A \), and

which are two time steps away from \( A \). With some imagination, one can see that the directional rela-

tionships between point \( A \) and the outer twelve points repeats itself around every point in the mesh, regardless of orientation of the inner four points, \( B, C, D, \) and \( E \).

Therefore, we take note of the following relationships, which may be derived in a manner similar to (7),

\[
V_l = \frac{1}{2} \left( \frac{z^{-1}}{1 + z^{-2}} \right) \left( V_k + \sum_{V_l} \right)
\]

where \( \Gamma \in \{ B, C, D, E \} \) and \( \gamma \in \{ 2, 8, 9 \} \), \( \gamma \in \{ 4, 5, 6, 7 \}, \gamma \in \{ 10, 11, 12 \} \), and \( \gamma \in \{ 1, 0, 7 \}. \)

Plugging (8) back into (7), we get an expression for \( V_k \) in terms of the junction velocities of the twelve junctions, \( V_l \):

\[
V_k = \frac{1}{4} \left( \frac{z^{-2}}{1 + z^{-2} + z^{-4}} \right) \sum_{V_l}
\]

To see that this partial difference equation approximates the 3D wave equation, we first mul-
tiply through by the denominator in (9), inverse \( \mathcal{Z} \)-transform, and gather all the terms onto the

left hand side. Then we view the equation as a continuous time and space expression of the wave function \( F(t, \vec{r}) \), if \( F(t, \vec{r}) \) is,

\[
\sum_{k=0}^{\infty} v(t - 2k \epsilon) \epsilon - \frac{1}{4} \sum_{k=0}^{\infty} v(t - \epsilon - 2k \epsilon + \epsilon \Delta \epsilon)
\]

and \( \epsilon \) is now the arbitrary spatial position of junction \( A \) and the \( \vec{r} \) represent the two-axis direc-
tional vectors from point \( A \) to the junction points marked 1 through 12 in Figure 9, respectively.

The unit time and space steps are defined as \( \epsilon \).

We may expand (10) in a four dimensional Taylor series about the point \( \vec{r} = (0, 0, 0) \) at time \( t = 0 \), replacing each term of (10) with something of the form,

\[
\sum_{n_1 = 0}^{\infty} \sum_{n_2 = 0}^{\infty} \sum_{n_3 = 0}^{\infty} \sum_{n_4 = 0}^{\infty} \frac{n_1 n_2 n_3 n_4}{n_1! n_2! n_3! n_4!} u_{n_1} u_{n_2} u_{n_3} u_{n_4}
\]

Collecting terms and computing the limit as the grid size shrinks reveals that

\[
\lim_{(2\epsilon)^2} F \left( \frac{x}{(2\epsilon)^2} \right) = u_{n_1} - \frac{1}{2} [u_{n_1} + u_{n_2} + u_{n_3}]
\]

Evidently the tetrahedral waveguide mesh is equivalent to an infinite difference approximation of the continuous 3D wave equation. The apparent wave speed is \( c = \sqrt{1/3} \), which is the num-

erically optimal speed in the Courant-Friedrichs-

Lewey sense [5]. (Incidentally, we found it conve-

nient to use the symbol manipulating feature of the mathematics processing language Mathematica to verify the algebra.)

To quantize dispersion error in the tetrahedral mesh, we apply a spectral transform analysis di-

rectly on the finite difference equation [5, 9]. Es-

sentially, we transform the difference equation into the frequency domain in both time and space, re-

placing spatial shifts with their corresponding spa-

tial linear phase terms. Then we observe how the spatial spectrum updates after one time sample.

With this information, we can determine how fast the various plane waves travel in the mesh at each frequency.

There can be no attenuation since the mesh is constructed from lossless scattering junc-

Therefore, the only departure from ideal behavior, aside from round-off error, is traveling-

wave dispersion.

We may now take the spatial Fourier transform of (9) and replace the spatial positions of the twelve outer junction points with their corresponding lin-

ear phase terms, \( V_k \rightarrow V_k e^{2\pi i k \Delta \epsilon} \), where \( \Delta \epsilon \) is the three-dimensional spatial frequency vector, to obtain the following quadratic expression in \( z^2 \):

\[
1 + b z^{-2} + z^{-4} = 0 \quad b \neq 1 - \frac{12}{4} \sum_{i=1}^{N_z^2} \Delta \epsilon^2 \times (13)
\]
Figure 7 compares the dispersion properties of the 3-port hexagonal and the 6-port triangular mesh structure. These mesh structures are not multiply-free; however, the triangular mesh seems to exhibit an optimal direction-independent dispersion. This is the underlying mesh used by Fontana and Rocchesso mentioned above [3]. Figure 8 shows a tricky variation to the 6-port triangular structure to achieve a multiply-free junction computation (although it requires one extra binary right shift). Here, two of the six waveguide segments connected at each junction are twice as thick as, i.e., twice the wave impedance of, the other four segments. This results in faster wave travel in one direction (much greater than that caused by dispersion error), but may be compensated for by resampling the spatial grid compressing the x-direction by a factor of $\sqrt{3}/3$. There is a similar trick making the 3D rectilinear 6-port mesh multiply-free, though it is rather heavy-handed in light of the tetrahedral alternative.

In a bounded mesh, wave speed dispersion results in a slight mistuning of the higher resonant modes. This mistuning can be adjusted by allpass filtering [8] and/or warping of the membrane boundary in a compensating manner. We note that the high frequency modes of a membrane become so dense that, in musical contexts, this error may not be psychoacoustically important. If higher accuracy is required, then an accordingly higher sampling rate may be used.

3 The Tetrahedral Difference

Rectilinear meshes compute finite difference approximation of the lossless wave equation [2, 10]. It is less obvious in the tetrahedral case. Figure 9 shows a small chunk of the tetrahedral mesh. We take the distance between adjacent junctions to be 1, and the junction point

![Figure 8: An isotropic Multiply-free 6-port Mesh](image)

Figure 9: Tetrahedral Structure Detail

marked A to lie at the origin of an $(x, y, z)$ cartesian coordinate system. We arrange the junctions $B(0, 2\sqrt{2}/3, 1/3)$, $C(\sqrt{2}/3, -\sqrt{2}/3, 1/3)$, $D(0, 0, -1)$, and $E(\sqrt{2}/3, -\sqrt{2}/3, 1/13)$, tetrahedrally about point $A(0, 0, 0)$. The line segments between these junction points represent bi-directional delay units as shown in Figure 10.

The equations describing the computation of the lossless 4-port scattering junctions are [3, 9, 10],

$$V_A = \frac{1}{2} \sum_{\Gamma} V_A^{\Gamma}$$

(1)

$$V_A^{\Gamma} = V_A - V_A^{\Gamma}$$

(2)

where $\Gamma$ ranges over the four junction points surrounding $A$, namely $\Gamma \in \{B, C, D, E\}$. $V_A^{\Gamma}$ represents the junction velocity at junction $A$. $V_A^{\Gamma}$ and $V_A^{\Gamma}$ represent the input and output signals, respectively, of junction $A$ in the direction of junction $\Gamma$.

Since the junctions are interconnected with bi-directional delay units, the input to junction $A$

![Figure 10: Bi-Directional Delay Unit](image)
4 Conclusions

Both the rectilinear and the tetrahedral 3D meshes have reasonable dispersion characteristics. And both model a wave speed of $c = \sqrt{\frac{1}{3}}$ space samples per time sample. We compute that the number of tetrahedrally arranged junctions required to fill a given volume is 35% less than that required for the rectilinear mesh; and the number of bi-directional delay units required for the tetrahedral mesh is 575% less than that required for the rectilinear mesh to fill the same given volume, thus saving substantial memory. Furthermore, the tetrahedral mesh is multiply-free and may be implemented efficiently in high-speed hardware. Applications in concert hall design, acoustical research, musical instrument synthesis, and reverberation are now practical.

References