and this is the result to be proved. If we replace $p$ by $p+1$, we find that

$$z^{p+1} \left( \frac{d}{dz} \right)^{p+1} \left( \frac{d}{dz} \right) \left( \frac{1}{z} \right) = \frac{1}{z^{p+1}} \left[ \left( \frac{d}{dz} \right)^{p} \frac{1}{z^{p+1}} \right].$$

When we transform (2) and (3) with the aid of (4) and (6), we see that the general solution of (1) is expressible in the following forms:

$$u = \frac{1}{z^{p+1}} \left( \frac{d}{dz} \right)^{p} \frac{\alpha e^{cz} + \beta e^{-cz}}{z^{p+1}},$$

$$u = \frac{1}{z^{p+1}} \left( \frac{d}{dz} \right)^{p+1} \frac{\alpha' e^{cz} + \beta' e^{-cz}}{z^{p+1}}.$$

The solutions of the equation

$$\frac{d^2 v}{dz^2} - \frac{2p}{z} \frac{dv}{dz} - c^2 v = 0,$$

[(3) of § 4·3], which correspond to (2), (3), (7) and (8) are

$$v = z^{p+1} \left( \frac{d}{dz} \right)^{p} \frac{\alpha e^{cz} + \beta e^{-cz}}{z},$$

$$v = z^{p+1} \left( \frac{d}{dz} \right)^{p} \frac{(\alpha' e^{cz} + \beta' e^{-cz})}{z},$$

$$v = \frac{1}{z} \left( \frac{d}{dz} \right)^{p} \frac{\alpha e^{cz} + \beta e^{-cz}}{z^{p+1}},$$

$$v = \frac{1}{z} \left( \frac{d}{dz} \right)^{p+1} \frac{\alpha' e^{cz} + \beta' e^{-cz}}{z^{p+1}}.$$


Similar symbolic solutions for the equation $\frac{d^2 v}{dz^2} - c^2 s^2 v = 0$ were discussed by Fields, *John Hopkins University Circulars*, vi. (1886—7), p. 29.

A transformation of the solution (9), due to Williamson, *Phil. Mag.* (4) xi. (1856), pp. 364—371, is

$$v = c^p \left( \frac{\partial}{\partial c} \right)^p \left( \alpha e^{cz} + \beta e^{-cz} \right).$$

This is derived from the equivalence of the operators $\frac{1}{c} \frac{\partial}{\partial c} + \frac{1}{c} \frac{\partial}{\partial c}$, when they operate on functions of $cz$.

We thus obtain the equivalence of the following operators

$$z^{p+1} \left( \frac{d}{dz} \right)^p \frac{1}{z} = (cz)^{2p+1} \left( \frac{\partial}{\partial (cz)} \right)^p \frac{1}{cz},$$

$$= (cz)^{2p+1} \left( \frac{\partial}{\partial (cz)} \right)^p \frac{1}{cz} = c^{2p+1} \left( \frac{\partial}{\partial (cz)} \right)^p \frac{1}{cz},$$

it being supposed that the operators operate on a function of $cz$; and Williamson's formula is then manifest.
4.7. Liouville's classification of elementary transcendental functions.

Before we give a proof of Liouville's general theorem (which was mentioned in §4.12) concerning the impossibility of solving Riccati's equation "in finite terms" except in the classical cases discovered by Daniel Bernoulli (and the limiting form of index -2), we shall give an account of Liouville's* theory of a class of functions known as elementary transcendental functions; and we shall introduce a convenient notation for handling such functions.

For brevity we write†

\[ l_1(z) = \log z, \quad l_n(z) = l(l(z)), \quad e_1(z) = e, \quad e_n(z) = e(e(z)), \quad s_1(z) = \int f(z) \, dz, \quad s_n(z) = s(s(z)), \quad s_n(z) = s(s_n(z)), \quad \ldots \]

A function of \( z \) is then said to be an elementary transcendental function‡ if it is expressible as an algebraic function of \( z \) and of functions of the types \( l, \phi(z), e, \psi(z), s, \chi(z) \), where the auxiliary functions \( \phi(z), \psi(z), \chi(z) \) are expressible in terms of \( z \) and of a second set of auxiliary functions, and so on; provided that there exists a finite number \( n \), such that the \( n \)th set of auxiliary functions are all algebraic functions of \( z \).

The order of an elementary transcendental function of \( z \) is then defined inductively as follows:

(I) Any algebraic function of \( z \) is of order zero.§

(II) If \( f_r(z) \) denotes any function of order \( r \), then any algebraic function of functions of the types

\[ l f_r(z), \quad e f_r(z), \quad s f_r(z), \quad f_r(z), \quad f_{r-1}(z), \ldots f_0(z) \]

(into which at least one of the first three enters) is said to be of order \( r + 1 \).

(III) Any function is supposed to be expressed as a function of the lowest possible order. Thus \( e l f_r(z) \) is to be replaced by \( f_r(z) \), and it is a function of order \( r \), not of order \( r + 2 \).

In connexion with this and the following sections, the reader should study Hardy, Orders of Infinity (Camb. Math. Tracts, no. 12, 1910). The functions discussed by Hardy were of a slightly more restricted character than those now under consideration, since, for his purposes, the symbol \( z \) is not required, and also, for his purposes, it is convenient to postulate the reality of the functions which he investigates.

It may be noted that Liouville did not study properties of the symbol \( s \) in detail, but merely remarked that it had many properties akin to those of the symbol \( \ell \).


† It is supposed that the integrals are all indefinite.

‡ "Une fonction finie explicite."

§ For the purposes of this investigation, irrational powers of \( z \), such as \( z^\pi \), of course must not be regarded as algebraic functions.
4.71. Liouville's first theorem* concerning linear differential equations.

The investigation of the character of the solution of the equation

\[ \frac{d^2 u}{dx^2} = u \chi(z), \]

in which \( \chi(z) \) is a transcendant of order \( \dagger \) \( n \), has been made by Liouville, who has established the following theorem:

If equation (1) has a solution which is a transcendant of order \( m + 1 \), where \( m > n \), then either there exists a solution of the equation which is of order \( \dagger \) \( n \), or else there exists a solution, \( u_1 \), of the equation expressible in the form

\[ u_1 = \phi_\mu(z) \cdot e^{f_\mu(z)}, \]

where \( f_\mu(z) \) is of order \( \mu \), and the order of \( \phi_\mu(z) \) does not exceed \( \mu \), and \( \mu \) is such that \( n \leq \mu \leq m \).

If the equation (1) has a solution of order \( m + 1 \), let it be \( f_{m+1}(z) \); then \( f_{m+1}(z) \) is an algebraic function of one or more functions of the types \( l f_m(z) \), \( sf_m(z) \), \( ef_m(z) \) as well as (possibly) of functions whose order does not exceed \( m \). Let us concentrate our attention on a particular function of one of the three types, and let it be called \( \theta \), \( \Sigma \) or \( \Theta \) according to its type.

(1) We shall first show how to prove that, if (1) has a solution of order \( m + 1 \), then a solution can be constructed which does not involve functions of the types \( \theta \) and \( \Sigma \).

For, if possible, let \( f_{m+1}(z) = F(z, \theta) \), where \( F \) is an algebraic function of \( \theta \); and any function of \( z \) (other than \( \theta \) itself) of order \( m + 1 \) which occurs in \( F \) is algebraically independent of \( \theta \).

Then it is easy to show that

\[ \frac{d^2 F}{dz^2} - F' \chi(z) = \frac{\partial^2 F}{\partial z^2} + \frac{2}{f_m(z)} \frac{df_m(z)}{dz} \frac{\partial^2 F}{\partial \theta \partial z} + \left[ \frac{1}{f_m(z)} \frac{df_m(z)}{dz} \right]^2 \frac{\partial^2 F}{\partial \theta^2} + \left[ \frac{d}{dz} \left( \frac{1}{f_m(z)} \frac{df_m(z)}{dz} \right) \right] \frac{\partial F}{\partial z} - F \chi(z), \]

it being supposed that \( z \) and \( \theta \) are the independent variables in performing the partial differentiations.

The expression on the right in (3) is an algebraic function of \( \theta \) which vanishes identically when \( \theta \) is replaced by \( l f_m(z) \). Hence it must vanish identically for all values of \( \theta \); for if it did not, the result of equating it to zero would express \( l f_m(z) \) as an algebraic function of transcendants whose orders do not exceed \( m \) together with transcendants of order \( m + 1 \) which are, ex hypothesi, algebraically independent of \( \theta \).

* * * * *


* This phrase is used as an abbreviation of "elementary transcendant function of order \( n \)."

† Null solutions are disregarded; if \( u \) were of order less than \( n \), then \( \frac{d^2 u}{u \cdot dx^2} \) would be of order less than \( n \), which is contrary to hypothesis.
In particular, the expression on the right of (3) vanishes when \( \theta \) is replaced by \( \theta + c \), where \( c \) is an arbitrary constant; and when this change is made the expression on the left of (3) changes into
\[
\frac{d^2F(z, \theta + c)}{dz^2} - F(z, \theta + c) \cdot \chi(z),
\]
which is therefore zero. That is to say
\[
\frac{d^2F(z, \theta + c)}{dz^2} = F(z, \theta + c) \cdot \chi(z) = 0.
\]
When we differentiate (4) partially with regard to \( c \), we find that
\[
\frac{\partial F(z, \theta + c)}{\partial c}, \quad \frac{\partial^2 F(z, \theta + c)}{\partial c^2}, \quad \ldots
\]
are solutions of (1) for all values of \( c \) independent of \( z \). If we put \( c = 0 \) after performing the differentiations, these expressions become
\[
\frac{\partial F(z, \theta)}{\partial \theta}, \quad \frac{\partial^2 F(z, \theta)}{\partial \theta^2}, \quad \ldots,
\]
which are consequently solutions of (1). For brevity they will be called \( F_\theta, F_{\theta\theta}, \ldots \).

Now either \( F \) and \( F_\theta \) form a fundamental system of solutions of (1) or they do not.

If they do not, we must have
\[
F_\theta = AF,
\]
where \( A \) is independent both of \( z \) and \( \theta \). On integration we find that
\[
F = \Phi e^{\alpha \theta},
\]
where \( \Phi \) involves transcendants (of order not exceeding \( m + 1 \)) which are algebraically independent of \( \theta \). But this is impossible because \( e^{\alpha \theta} \) is not an algebraic function of \( \theta \); and therefore \( F \) and \( F_\theta \) form a fundamental system of solutions of (1).

Hence \( F_{\theta\theta} \) is expressible in terms of \( F \) and \( F_\theta \) by an equation of the form
\[
F_{\theta\theta} = AF_\theta + BF,
\]
where \( A \) and \( B \) are constants. Now this may be regarded as a linear equation in \( \theta \) (with constant coefficients) and its solution is
\[
F = \Phi_1 e^{\alpha \theta} + \Phi_2 e^{\beta \theta} \text{ or } F = e^{\alpha \theta} \{ \Phi_1 + \Phi_2 \theta \},
\]
where \( \Phi_1 \) and \( \Phi_2 \) are functions of the same nature as \( \Phi \), while \( \alpha \) and \( \beta \) are the roots of the equation
\[
x^2 - Ax - B = 0.
\]
The only value of \( F \) which is an algebraic function of \( \theta \) is obtained when \( \alpha = \beta = 0 \); and then \( F \) is a linear function of \( \theta \).

Similarly, if \( f_{m+1}(z) \) involves a function of the type \( \Phi \), we can prove that it must be a linear function of \( \Phi \).

* Since \( F \) must involve \( \theta \), \( F_\theta \) cannot be identically zero.
It follows that, in so far as $f_{m+1}(z)$ involves functions of the types $\theta$ and $\Psi$, it involves them linearly, so that we may write

$$f_{m+1}(z) = \sum \theta_1(z) \theta_2(z) \cdots \theta_p(z) \cdot \Psi_1(z) \Psi_2(z) \cdots \Psi_q(z),$$

where the functions $\Psi_{p,q}(z)$ are of order $m+1$ at most, and the only functions of order $m+1$ involved in them are of the type $\Theta$.

Take any one of the terms in $f_{m+1}(z)$ which is of the highest degree, qua function of $\theta_1, \theta_2, \ldots, \Psi_1, \Psi_2, \ldots$, and let it be

$$\theta_1(z) \theta_2(z) \cdots \theta_p(z) \cdot \Psi_1(z) \Psi_2(z) \cdots \Psi_q(z).$$

Then, by arguments resembling those previously used, it follows that

$$\left[ \frac{\partial}{\partial \theta_1} \frac{\partial}{\partial \theta_2} \cdots \frac{\partial}{\partial \theta_p} \frac{\partial}{\partial \Psi_1} \frac{\partial}{\partial \Psi_2} \cdots \frac{\partial}{\partial \Psi_q} \right] f_{m+1}(z)$$

is a solution of (1); i.e. $\Psi_{p,q}(z)$ is a solution of (1).

But $\Psi_{p,q}(z)$ is either a function of order not exceeding $m$, or else it is a function of order $m+1$ which involves functions of the type $\Theta$ and not of the types $\theta$ and $\Psi$.

In the former case, we repeat the process of reduction to functions of lower order, and in the latter case we see that some solution of the equation is an algebraic function of functions of the type $\Theta$.

We have therefore proved that, if (1) has a solution which is a transcendant of order greater than $n$, then either it has a solution of order $n$ or else it has a solution which is an algebraic function of functions of the type $e_{\nu}(z)$ and $\phi_{\mu}(z)$, where $f_{\nu}(z)$ is of order $\mu$ and $\phi_{\mu}(z)$ is of an order which does not exceed $\mu$.

(II) We shall next prove that, whenever (1) has a solution which is a transcendant of order greater than $n$, then it has a solution which involves the transcendant $e_{\nu}(z)$ only in having a power of it as a factor.

We concentrate our attention on a particular transcendant $\Theta$ of the form $e_{\nu}(z)$, and then the postulated solution may be written in the form $G(z, \Theta)$, where $G$ is an algebraic function of $\Theta$; and any function (other than $\Theta$ itself) of order $\mu + 1$ which occurs in $G$ is algebraically independent of $\Theta$.

Then it is easy to show that

$$\frac{\partial^2 G}{\partial z^2} - G \cdot \chi(z) = \frac{\partial^2 G}{\partial \theta^2} + 2i \Psi_1' z \frac{\partial^2 G}{\partial \psi \partial \Theta} + (\Theta f_{\nu}'(z))^2 \frac{\partial^2 G}{\partial \Theta^2}$$

$$+ \Theta \left[ f_{\nu}''(z) + (f_{\nu}'(z))^2 \right] \frac{\partial G}{\partial \Theta} - G \cdot \chi(z).$$

The expression on the right is an algebraic function of $\Theta$ which vanishes when $\Theta$ is replaced by $e_{\nu}(z)$, and so it vanishes identically, by the arguments used in (I). In particular it vanishes when $\Theta$ is replaced by $c \Theta$, where $c$ is independent of $z$. But its value is then

$$\frac{\partial^2 G}{\partial z^2} - G \cdot \chi(z),$$
so that

\[
\frac{d^2 G (z, c \Theta)}{dz^2} - G (z, c \Theta) \cdot \chi (z) = 0.
\]

When we differentiate this with regard to \(c\), we find that

\[
\frac{\partial G (z, c \Theta)}{\partial c}, \quad \frac{\partial^2 G (z, c \Theta)}{\partial c^2}, \quad \ldots
\]

are solutions of (1) for all values of \(c\) independent of \(z\). If we put \(c = 1\), these expressions become

\[
\Theta \frac{\partial G (z, \Theta)}{\partial \Theta}, \quad \Theta^2 \frac{\partial^2 G (z, \Theta)}{\partial \Theta^2}, \quad \ldots
\]

Hence, by the reasoning used in (1), we have \(\Theta G \Theta = A G\) or else

\[
\Theta^2 G \Theta = A \Theta G + BG,
\]

where \(A\) and \(B\) are constants.

In the former case \(G = \Phi \Theta^4\), and in the latter \(G\) has one of the values

\[
\Phi_1 \Theta^\gamma + \Phi_2 \Theta^\delta \quad \text{or} \quad \Theta^\gamma \left[ \Phi_1 + \Phi_2 \log \Theta \right] = \Theta^\gamma \left[ \Phi_1 + \Phi_2 f_\mu (z) \right],
\]

where \(\Phi, \Phi_1, \Phi_2\) are functions of \(z\) of order \(\mu + 1\) at most, any functions of order \(\mu + 1\) which are involved being algebraically independent of \(\Theta\); while \(\gamma\) and \(\delta\) are the roots of the equation

\[
x (x - 1) - Ax - B = 0.
\]

In any case, \(G\) either contains \(\Theta\) only by a factor which is a power of \(\Theta\) or else \(G\) is the sum of two expressions which contain \(\Theta\) only in that manner. In the latter case *,

\[
G (z, c \Theta) - c^4 G (z, \Theta)
\]

is a solution of (1) which contains \(\Theta\) only by a factor which is a power of \(\Theta\).

By repetitions of this procedure, we see that, if \(\Theta_1, \Theta_2, \ldots \Theta_r\) are all the transcendents of order \(\mu + 1\) which occur in the postulated solution, we can derive from that solution a sequence of solutions of which the \(s\)th contains \(\Theta_1, \Theta_2, \ldots \Theta_s\) only by factors which are powers of \(\Theta_1, \Theta_2, \ldots \Theta_s\); and the \(r\)th member of the sequence consequently consists of a product of powers of \(\Theta_1, \Theta_2, \ldots \Theta_r\) multiplied by a transcendant which is of order \(\mu\) at most; this solution is of the form

\[
\phi_\mu (z) \exp \left\{ \sum_{s=1}^{r} \gamma_s \log \Theta_s \right\},
\]

which is of the form \(\phi_\mu (z) \cdot e^{f_\mu (z)}\).

* If \(\Phi_1\) is not identically zero; if it is, then \(\Phi_2 \Theta^\delta\) is a solution of the specified type.
4.72. **Liouville's second theorem concerning linear differential equations.**

We have just seen that, if the equation

\[ \frac{d^2 u}{dz^2} = u \chi(z) \]  

[in which \( \chi(z) \) is of order \( n \)] has a solution which is an elementary transcendant of order greater than \( n \), then it must have a solution of the form

\[ \phi_\mu(z) \eta_\mu(z), \]

where \( \mu \geq n \). If the equation has more than one solution of this type, let a solution for which \( \mu \) has the smallest value be chosen, and let it be called \( u_1 \).

Liouville's theorem, which we shall now prove, is that, for this solution, the order of \( d (\log u_1)/dz \) is equal to \( n \).

Let

\[ \frac{d \log u_1}{dz} \equiv t, \]

and then \( t \) is of order \( \mu \) at most; let the order of \( t \) be \( N \), where \( N \leq \mu \).

If \( N = n \), the theorem required is proved. If \( N > n \), then the equation satisfied by \( t \), namely

\[ \frac{dt}{dz} + \theta = \chi(z), \]

has a solution whose order \( N \) is greater than \( n \).

Now \( t \) is an algebraic function of at least one transcendant of the types \( \eta f_{N-1}(z), \eta f_N(z), \eta f_{N-1}(z) \) and (possibly) of transcendants whose order does not exceed \( N-1 \). We call the first three transcendants \( \theta, \Theta, \Theta \) respectively.

If \( t \) contains more than one transcendant of the type \( \theta \), we concentrate our attention on a particular function of this type, and we write

\[ t = F(z, \theta). \]

By arguments resembling those used in § 4.71, we find that, if \( N > n \), then

\[ F(z, \theta + c) \]

is also a solution of (2). The corresponding solution of (1) is

\[ \exp \int F(z, \theta + c) \, dz, \]

and this is a solution for all values of \( c \) independent of \( z \). Hence, by differentiation with respect to \( c \), we find that the function \( u_0 \) defined as

\[ \left[ \frac{d}{dc} \left( \exp \int F(z, \theta + c) \, dz \right) \right]_{c=0} \]

is also a solution of (1); and we have

\[ u_0 = u_1 \int F_\theta \, dz, \]

so that

\[ u_1 \frac{du_0}{dz} - u_0 \frac{du_1}{dz} = u_0^2 F_\theta. \]
But the Wronskian of any two solutions of (1) is a constant*; and so
\[ u_1^r F_0 = C, \]
where \( C \) is a constant.

If \( C = 0 \), \( F \) is independent of \( \theta \), which is contrary to hypothesis; so \( C \neq 0 \), and
\[ u_1 = \sqrt{C / F_0}. \]
Hence \( u_1 \) is an algebraic function of \( \theta \); and similarly it is an algebraic function of all the functions of the types \( \theta \) and \( \mathcal{S} \) which occur in \( t \).

Next consider any function of the type \( \Theta \) which occurs in \( t \); we write
\[ t = G(z, \Theta), \]
and, by arguments resembling those used in \( \S \, 4.71 \) and those used earlier in this section, we find that the function \( u_2 \) defined as
\[ \frac{\partial}{\partial c} \left[ \exp \left\{ \int G(z, c\Theta) \, dz \right\} \right]_{c=1} \]
is a solution of (1); and we have
\[ u_2 = u_1 \int \Theta G_\Theta \, dz, \]
so that
\[ u_1 \frac{du_2}{dz} - u_2 \frac{du_1}{dz} = u_1^2 \Theta G_\Theta. \]

This Wronskian is a constant, \( C_1 \), and so
\[ u_1 = \sqrt[2]{C_1 / (\Theta G_\Theta)}. \]
Consequently \( u_1 \) is an algebraic function, not only of all the transcendents of the types \( \theta \) and \( \mathcal{S} \), but also of those of type \( \Theta \) which occur in \( t \); and therefore \( u_1 \) is of order \( N \). This is contrary to the hypothesis that \( u_1 \) is of order \( \mu + 1 \), where \( \mu \geq N \), if \( N > n \).

The contradiction shows that \( N \) cannot be greater than \( n \); hence the order of \( d(\log u_1)/dz \) is \( n \). And this is the theorem to be established.

4.73. Liouville's theorem† that Bessel's equation has no algebraic integral.

We shall now show that the equation
\[ z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2) y = 0 \]
has no integral (other than a null-function) which is an algebraic function of \( z \).

We first reduce the equation to its normal form
\[ \frac{d^2 u}{dz^2} + \left( 1 - \frac{p(p+1)}{z^2} \right) u = 0, \]
by writing
\[ y = uz^p, \quad p = \pm \nu - \frac{1}{2}. \]

* See e.g. Forsyth, Treatise on Differential Equations (1914), § 65.
† Journal de Math. iv. (1839), pp. 429--435; vi. (1841), pp. 4--7. Liouville's first investigation was concerned with the general case in which \( \chi(z) \) is any polynomial; the application (with various modifications) to Bessel's equation was given in his later paper, Journal de Math. vi. (1841), pp. 1--13, 86.
This is of the form
\[ \frac{d^2 u}{dz^2} = u \chi(z), \]
where
\[ \chi(z) = \frac{p(p+1)}{z} - 1. \]

If possible, let Bessel's equation have an algebraic integral; then (1) also has an algebraic integral. Let the equation which expresses this integral, \( u \), as an algebraic function of \( z \) be
\[ \mathcal{A}(u, z) = 0, \]
where \( \mathcal{A} \) is a polynomial both in \( u \) and in \( z \); and it is supposed that \( \mathcal{A} \) is irreducible*.

Since \( u \) is a solution of (1) we have
\[ \mathcal{A}_{00} u^2 - 2 \mathcal{A}_{01} u + \mathcal{A}_{1} u + \mathcal{A}_{2} u^2 + \mathcal{A}_{3} u^3 + \mathcal{A}_{4} u^4 \chi(z) = 0. \]

The equations (3) and (4) have a common root, and hence all the roots of (3) satisfy (4).

For, if not, the left-hand sides of (3) and (4) (qua functions of \( u \)) would have a highest common factor other than \( \mathcal{A} \) itself, and this would be a polynomial in \( u \) and in \( z \). Hence \( \mathcal{A} \) would be reducible, which is contrary to hypothesis.

Let all the roots of (3) be \( u_1, u_2, \ldots, u_M \). Then, if \( s \) is any positive integer,
\[ u_1^s + u_2^s + \ldots + u_M^s \]
is a rational function of \( z \); and there is at least one value of \( s \) not exceeding \( M \) for which this sum is not zero†.

Let any such value of \( s \) be taken, and let
\[ W_0 = \sum_{m=1}^{M} u_m^s, \]
Also let
\[ W_r = s(s-1) \ldots (s-r+1) \sum_{m=1}^{M} u_m^{s-r} (\frac{du_m}{dz})^r, \]
where \( r = 1, 2, \ldots, s \). Since \( u_1, u_2, \ldots, u_M \) are all solutions‡ of (1), it is easy to prove that
\[ \frac{dW_0}{dz} = W_1, \]
\[ \frac{dW_r}{dz} = W_{r+1} + r(s-r+1) \chi(z) W_{r-1}, \quad (r = 1, 2, \ldots, s-1) \]
\[ \frac{dW_s}{dz} = s \chi(z) W_{s-1}. \]

* That is to say, \( \mathcal{A} \) has no factors which are polynomials in \( u \) or in \( z \) or in both \( u \) and \( z \).
† If not, all the roots of (3) would be zero.
‡ Because (4) is satisfied by all the roots of (3), qua equation in \( u \).
Since \( W_\omega \) is a rational function of \( \omega \), it is expressible in partial fractions, so that
\[
W_\omega = \sum_{n=-\infty}^{\lambda} A_n \omega^n + \sum_{n,q} \frac{B_{n,q}}{(\omega - a_q)^n},
\]
where \( A_n \) and \( B_{n,q} \) are constants, \( \kappa \) and \( \lambda \) are integers, \( n \) assumes positive integral values only in the last summation and \( a_q \neq 0 \).

Let the highest power of \( 1/(\omega - a_q) \) which occurs in \( W_\omega \) be \( 1/(\omega - a_q)^r \).

It follows by an easy induction from (5) and (6) that the highest power of \( 1/(\omega - a_q) \) in \( W_\omega \) is \( 1/(\omega - a_q)^{r+\tau} \), where \( r = 1, 2, \ldots, s \).

Hence there is a higher power on the left of (7) than on the right. This contradiction shews that there are no terms of the type \( B_{n,q} (\omega - a_q)^{-\kappa} \) in \( W_\omega \) and so
\[
W_\omega = \sum_{n=-\infty}^{\lambda} A_n \omega^n.
\]

We may now assume that \( A_\lambda \neq 0 \), because this expression for \( W_\omega \) must have a last term if it does not vanish identically.

From (5) and (6) it is easy to see that the terms of highest degree in \( \omega \) which occur in \( W_\omega, W_1, W_2, W_3, \ldots \) are
\[
A_\lambda \omega^{\lambda}, \quad \lambda A_\lambda \omega^{\lambda-1}, \quad A_\lambda s \omega^{\lambda}, \quad \lambda A_\lambda (3s - 2) \omega^{\lambda-1}, \ldots
\]

By a simple induction it is possible to shew that the term of highest degree in \( W_{s+1} \) is
\[
A_\lambda \omega^{\lambda}, 1, 3 \ldots (2r - 1), s, (s - 2) \ldots (s - 2r + 2).
\]

An induction of a more complicated nature is then necessary to shew that the term of highest degree in \( W_{s+1} \) is
\[
\lambda A_\lambda \omega^{\lambda-1} 2, 4 \ldots (2r), (s - 1), (s - 3) \ldots (s - 2r + 1) F_1 \left( \frac{1}{2}, -\frac{1}{2} s; \frac{1}{2} - \frac{1}{2} s; 1 \right)_{r+1},
\]
where the suffix \( r + 1 \) indicates that the first \( r + 1 \) terms only of the hypergeometric series are to be taken.

If \( s \) is odd, the terms of highest degree on the left and right of (7) are of degrees \( \lambda - 2 \) and \( \lambda \) respectively, which is impossible. Hence \( W_\omega \) vanishes whenever \( s \) is odd.

When \( s \) is even, the result of equating coefficients of \( \omega^{\lambda-1} \) in (7) is
\[
\lambda A_\lambda s! = -\lambda A_\lambda s! \cdot F_1 \left( \frac{1}{2}, -\frac{1}{2} s; \frac{1}{2} - \frac{1}{2} s; 1 \right)_{r+1}.
\]
That is to say
\[
\lambda A_\lambda s! F_1 \left( \frac{1}{2}, -\frac{1}{2} s; \frac{1}{2} - \frac{1}{2} s; 1 \right) = 0,
\]
and so, by Vandermonde's theorem,
\[
\lambda A_\lambda \frac{s!}{1 \cdot 3 \cdot 5 \ldots (s - 1)} = 0.
\]

The expression on the left vanishes only when \( \lambda \) is zero.\]
We have therefore proved that, when \( s \) is odd, \( W_0 \) vanishes, and that, when \( s \) is even, \( W_0 \) is expressible in the form

\[
\sum_{n=0}^{\infty} A_{n,s} x^{-n},
\]

where \( A_{n,s} \) does not vanish.

From Newton's theorem which expresses the coefficients in an equation in terms of the sums of powers of the roots, it appears that \( M \) must be even, and that the equation \( \mathcal{A}(u, z) = 0 \) is expressible in the form

\[
M + \sum_{r=1}^{M} \mathcal{P}_r(1/z) = 0,
\]

where the functions \( \mathcal{P}_r \) are polynomials in \( 1/z \).

When we solve (8) in a series of ascending powers of \( 1/z \), we find that each of the branches of \( u \) is expressible in the form

\[
\sum_{m=0}^{\infty} c_m z^{-m/n},
\]

where \( n \) is a positive integer and, in the case of one branch at least, \( c_0 \) does not vanish because the constant terms in the functions \( \mathcal{P}_r \) are not all zero. And the series which are of the form

\[
\sum_{m=0}^{\infty} c_m z^{-m/n}
\]

are convergent* for all sufficiently large values of \( z \).

When we substitute the series into the left-hand side of (1), we find that the coefficient of the constant term in the result is \( c_0 \), and so, for every branch, \( c_0 \) must be zero, contrary to what has just been proved. The contradiction thus obtained shews that Bessel's equation has no algebraic integral.

4·74. On the impossibility of integrating Bessel's equation in finite terms.

We are now in a position to prove Liouville's theorem† that Bessel's equation for functions of order \( \nu \) has no solution (except a null-function) which is expressible in finite terms by means of elementary transcendental functions, if \( 2\nu \) is not an odd integer.

As in § 4·73, we reduce Bessel's equation to its normal form

\[
\frac{d^2 u}{dz^2} = u \chi(z),
\]

where \( \chi(z) = -1 + p(p + 1)/z^2 \) and \( p = \pm \nu - \frac{1}{2} \).

Now write \( d(\log u)/dz = t \), and we have

\[
\frac{dt}{dz} + \nu + 1 - \frac{p(p + 1)}{z^2} = 0.
\]

* Goursat, Cours d'Analyse, ii. (Paris, 1911), pp. 273—281. Many treatises tacitly assume the convergence of a series derived in this manner from an algebraic equation.

† Journal de Math. vi. (1841), pp. 1—13, 36.
Since \( \chi(z) \) is of order zero, it follows from § 472 that, if Bessel’s equation has an integral expressible in finite terms, then (2) must have a solution which is of order zero, i.e. it must have an algebraic integral.

If (2) has an algebraic integral, let the equation which expresses this integral, \( t \), as an algebraic function of \( z \), be

\[
(3) \quad \mathcal{A}(t, z) = 0,
\]

where \( \mathcal{A} \) is an irreducible polynomial in \( t \) and \( z \).

Since \( t \) is a solution of (2), we have

\[
(4) \quad \mathcal{A}_z + [\chi(z) - t^\prime] \mathcal{A}_t = 0.
\]

As in the corresponding analysis of § 473, all the branches of \( t \) satisfy (4).

First suppose that there are more than two branches of \( t \), and let three of them be called \( t_1, t_2, t_3 \), the corresponding values of \( u \) (defined as \( \exp \int t \, dz \)) being \( u_1, u_2, u_3 \). These functions are all solutions of (1) and so the Wronskians

\[
\frac{u_2 du_3}{dz} - \frac{u_3 du_2}{dz}, \quad \frac{u_3 du_1}{dz} - \frac{u_1 du_3}{dz}, \quad \frac{u_1 du_2}{dz} - \frac{u_2 du_1}{dz}
\]

are constants, which will be called \( C_1, C_2, C_3 \).

Now it is easy to verify that

\[
C_1 = u_2 \frac{du_3}{dz} - u_3 \frac{du_2}{dz} = u_2 u_3 (t_3 - t_2);
\]

and \( t_3 - t_2 \) is not zero, because, if it were zero, the equation (3) would have a pair of equal roots, and would therefore be reducible.

Hence \( C_1 \neq 0 \), and so

\[
u_2 u_3 = C_1 / (t_3 - t_2).
\]

Therefore \( u_2 u_3 \) (and similarly \( u_1 u_3 \) and \( u_1 u_2 \)) is an algebraic function of \( z \).

But

\[
u_1 = \sqrt{\frac{u_2 u_1 \cdot u_1 u_3}{u_2 u_3}},
\]

and therefore \( u_1 \) is an algebraic function of \( z \). This, as we have seen in §473, cannot be the case, and so \( t \) has not more than two branches.

Next suppose that \( t \) has two branches, so that \( \mathcal{A}(t, z) \) is quadratic in \( t \). Let the branches be \( U \pm \sqrt{V} \), where \( U \) and \( V \) are rational functions of \( z \). By substituting in (2) we find that

\[
(5) \quad \begin{cases} U' + U^* + V = \chi(z), \\ V' + 4UV = 0. \end{cases}
\]

Let \( V \) be factorised so that

\[
V = A z^\lambda \Pi (z - a_q) \nu_q,
\]

where \( A \) is constant, \( \kappa_q \) and \( \lambda \) are integers, and \( \kappa_q \) and \( a_q \) are not zero.
From the second member of (5) it follows that

\[ U = -\frac{\lambda}{4x} - \sum \frac{\kappa_q}{4} \frac{1}{(z - a_q)}, \]

and then by substituting into the first member of (5) we have

\[
\frac{\lambda}{4x} + \sum \frac{\kappa_q}{4(\sigma - a_q)^{\sigma}} + \frac{1}{4} \frac{\kappa_q}{(\sigma - a_q)^{\sigma}} + \lambda \sum \frac{1}{\sigma} \Pi (z - a_q)^{\sigma} \chi (x) \equiv 0. \tag{6}
\]

Now consider the principal part of the expression on the left near \( a_q \). It is evident that none of the numbers \( \kappa_q \) can be less than \(-2\), and, if any one of them is greater than \(-2\) it must satisfy the equation

\[ \kappa_q^2 + 4\kappa_q = 0, \]

so that \( \kappa_q \) is 0 or \(-4\), which are both excluded from consideration. Hence all the numbers \( \kappa_q \) are equal to \(-2\).

Again, if we consider the principal part near \( \infty \), we see that the highest power in \( V \) must cancel with the \(-1\) in \( \chi (x) \), so that \( \lambda = -\sum \kappa_q \).

It follows that \( \sqrt{V} \) is rational, and consequently \( \mathcal{A} (t, z) \) is reducible, which is contrary to hypothesis.

Hence \( t \) cannot have as many as two branches and so it must be rational.

Accordingly, let the expression for \( t \) in partial fractions be

\[ t = \sum \frac{A_n z^n + B_{n, q} (z - a_q)^n}{(z - a_q)^n}, \]

where \( A_n \) and \( B_{n, q} \) are constants, \( \kappa \) and \( \lambda \) are integers, \( n \) assumes positive values only in the last summation and \( a_q \neq 0 \).

If we substitute this value of \( t \) in (2) we find that

\[ \sum \frac{nA_n z^n - \sum \frac{nB_{n, q}}{(z - a_q)^{n+1}}} + \frac{1}{(z - a_q)^{n+1}} \equiv 0. \]

If we consider the principal part of the left-hand side near \( a_q \), we see that \( 1/(z - a_q) \) cannot occur in \( t \) to a higher power than the first and that

\[ B_{n, q} = B_{n, q} = 0, \]

so that

\[ B_{n, q} = 1. \]

Similarly, if we consider the principal parts near 0 and \( \infty \), we find that

\[ \kappa = 1, \quad (A_{-\kappa})^2 - A_{-\kappa} = p (p + 1); \quad \lambda = 0, \quad A_{-\kappa} = -1. \]

Since \( p = \pm \nu - \frac{1}{2} \), we may take \( A_{-1} = -p \) without loss of generality.

It then follows that

\[ u = z^{-p} e^{\pm iz} \Pi (z - a_q). \]

Accordingly, if we replace \( u \) by \( z^{-p} e^{\pm iz} w \) in (1), we see that the equation

\[
\frac{d^2 w}{dz^2} + 2 \left( \frac{p}{z} \right) \frac{dw}{dz} + \frac{2ip}{z} w = 0 \tag{7}
\]

must have a solution which is a polynomial in \( z \), and the constant term in this polynomial does not vanish.
When we substitute $\Sigma c_ne^{nm}$ for $w$ in (7) we find that the relation connecting successive coefficients is

$$m(m - 2p - 1)c_m + 2i\mu_{m-1}(m - p - 1) = 0,$$

and so the series for $w$ cannot terminate unless $m - p - 1$ can vanish, i.e. unless $p$ is zero or a positive integer.

Hence the hypothesis that Bessel's equation is soluble in finite terms leads of necessity to the consequence that one of the numbers $\pm \nu - \frac{1}{2}$ is zero or a positive integer; and this is the case if, and only if, $2\nu$ is an odd integer.

Conversely we have seen (§ 3.4) that, when $2\nu$ is an odd integer, Bessel's equation actually possesses a fundamental system of solutions expressible in finite terms. The investigation of the solubility of the equation is therefore complete.

Some applications of this theorem to equations of the types discussed in § 4.3 have been recorded by Lebesgue, Journal de Math. xl. (1846), pp. 338—340.

4.75. On the impossibility of integrating Riccati's equation in finite terms.

By means of the result just obtained, we can discuss Riccati's equation

$$\frac{dy}{dz} = \alpha y^n + by^2$$

with a view to proving that it is, in general, not integrable in finite terms.

It has been seen (§ 4.21) that the equation is reducible to

$$\frac{d^2u}{d\xi^2} - \alpha^2 \xi^{2q-2} u = 0,$$

where $n = 2q - 2$; and, by § 4.3, the last equation is reducible to Bessel's equation for functions of order $1/(2q)$ unless $q = 0$.

Hence the only possible cases in which Riccati's equation is integrable in finite terms are those in which $q$ is zero or $1/q$ is an odd integer; and these are precisely the cases in which $n$ is equal to $-2$ or to

$$-\frac{4m}{2m \pm 1}. \quad (m = 0, 1, 2, \ldots)$$

Consequently the only cases in which Riccati's equation is integrable in finite terms are the classical cases discovered by Daniel Bernoulli (cf. § 4.11) and the limiting case discussed after the manner of Euler in § 4.12.

This theorem was proved by Liouville, Journal de Math. vi. (1841), pp. 1—13. It seems impossible to establish it by any method which is appreciably more brief than the analysis used in the preceding sections.
4.8. Solutions of Laplace's equation.

The first appearance in analysis of the general Bessel coefficient has been seen (§ 1.3) to be in connexion with an equation, equivalent to Laplace's equation, which occurs in the problem of the vibrations of a circular membrane.

We shall now show how Bessel coefficients arise in a natural manner from Whittaker's* solution of Laplace's equation

\[ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \]

The solution in question is

\[ V = \int_{-\pi}^{\pi} f(z + i\cos u + iy \sin u, u) du, \]

in which \( f \) denotes an arbitrary function of the two variables involved.

In particular, a solution is

\[ \int_{-\pi}^{\pi} e^{k(z + i\cos u + iy \sin u)} \cos m\phi \, du, \]

in which \( k \) is any constant and \( m \) is any integer.

If we take cylindrical-polar coordinates, defined by the equations

\[ x = \rho \cos \phi, \quad y = \rho \sin \phi, \]

this solution becomes

\[ e^{kz} \int_{-\pi}^{\pi} e^{ik\rho \cos (u - \phi)} \cos m\phi \, du = e^{kz} \int_{-\pi}^{\pi} e^{ik\rho \cos u} \cos m(v + \phi) \, dv, \]

\[ = 2e^{kz} \int_{0}^{\pi} e^{ik\rho \cos u} \cos mv \cos m\phi \, dv, \]

\[ = 2\pi i^m e^{kz} \cos m\phi \cdot J_m(k\rho), \]

by § 2.2. In like manner a solution is

\[ \int_{-\pi}^{\pi} e^{k(z + i\cos u + iy \sin u)} \sin m\phi \, du, \]

and this is equal to \( 2\pi i^m e^{kz} \sin m\phi \cdot J_m(k\rho) \). Both of these solutions are analytic near the origin.

Again, if Laplace's equation be transformed† to cylindrical-polar coordinates, it is found to become

\[ \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0; \]

and a normal solution of this equation of which $e^{ikz}$ is a factor must be such that
\[ \frac{1}{V} \frac{\partial^2 V}{\partial \phi^2} \]
is independent of $\phi$, and, if the solution is to be one-valued, it must be equal to $-m^2$ where $m$ is an integer. Consequently the function of $\rho$ which is a factor of $V$ must be annihilated by
\[ \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + \left( k^2 - m^2 \right) \frac{1}{\rho^2}, \]
and therefore it must be a multiple of $J_m(k\rho)$ if it is to be analytic along the line $\rho = 0$.

We thus obtain anew the solutions
\[ e^{ikz} \frac{\cos m\phi}{\sin} \cdot J_m(k\rho). \]

These solutions have been derived by Hobson* from the solution $e^{ik\theta} J_0(kr)$ by Clerk Maxwell’s method of differentiating harmonics with respect to axes.

Another solution of Laplace’s equation involving Bessel functions has been obtained by Hobson (ibid. p. 447) from the equation in cylindrical-polar coordinates by regarding $\partial/\partial z$ as a symbolic operator. The solution so obtained is
\[ \cos \frac{m\phi}{\sin} \cdot \mathcal{Q}_m \left( \rho \frac{d}{dz} \right) f(z), \]
where $f(z)$ is an arbitrary function; but the interpretation of this solution when $\mathcal{Q}_m$ involves a function of the second kind is open to question. Other solutions involving a Bessel function of an operator acting on an arbitrary function have been given by Hobson, Proc. London Math. Soc. xxiv. (1893), pp. 55—67; xxvi (1895), pp. 492—494.

481. Solutions of the equations of wave motions.

We shall now examine the equation of wave motions
\[ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2}, \]
in which $t$ represents the time and $c$ the velocity of propagation of the waves, from the same aspect.

Whittaker’s† solution of this equation is
\[ V = \int_{-\infty}^{\infty} \int_{0}^{\infty} f(x \sin u \cos v + y \sin u \sin v + z \cos u + ct, u, v) \, du \, dv, \]
where $f$ denotes an arbitrary function of the three variables involved.

In particular, a solution is
\[ V = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{ikz \sin u \cos v + y \sin u \sin v + z \cos u + ct} F(u, v) \, du \, dv, \]
where $F$ denotes an arbitrary function of $u$ and $v$.

The physical importance of this particular solution lies in the fact that it is the general solution in which the waves all have the same frequency $\omega c$.

Now let the polar coordinates of $(x, y, z)$ be $(r, \theta, \phi)$, and let $(\omega, \psi)$ be the angular coordinates of the direction $(u, v)$ referred to new axes for which the polar axis is the direction $(\theta, \phi)$ and the plane $\psi = 0$ passes through the $z$-axis. The well-known formulae of spherical trigonometry then show that

$$\cos \omega = \cos \theta \cos u + \sin \theta \sin u \cos (v - \phi),$$

$$\sin u \sin (v - \phi) = \sin \omega \sin \psi.$$

Now take the arbitrary function $F(u, v)$ to be $S_n(u, v) \sin u$, where $S_n$ denotes a surface harmonic in $(u, v)$ of degree $n$; we may then write

$$S_n(u, v) = S_n(\theta, \phi; \omega, \psi),$$

where $S_n$ is a surface harmonic* in $(\omega, \psi)$ of degree $n$.

We thus get the solution

$$V_n = e^{i\omega t} \int_{-\pi}^{\pi} e^{i\kappa \cos \omega} S_n(\theta, \phi; \omega, \psi) \sin \omega d\omega d\psi.$$

Since $S_n$ is a surface harmonic of degree $n$ in $(\omega, \psi)$, we may write

$$S_n(\theta, \phi; \omega, \psi) = A_n(\theta, \phi) P_n(\cos \omega)$$

$$+ \sum_{m=1}^{n} \{ A_n^{(m)}(\theta, \phi) \cos m \psi + B_n^{(m)}(\theta, \phi) \sin m \psi \} P_m^n(\cos \omega),$$

where $A_n(\theta, \phi), A_n^{(m)}(\theta, \phi)$ and $B_n^{(m)}(\theta, \phi)$ are independent of $\omega$ and $\psi$.

Performing the integration with respect to $\psi$, we get

$$V_n = 2\pi e^{i\omega t} A_n(\theta, \phi) \int_{0}^{\pi} e^{i\kappa \cos \omega} P_n(\cos \omega) \sin \omega d\omega$$

$$= (2\pi)^{\frac{3}{2}} e^{i\omega t} \frac{J_{n+\frac{1}{2}}(kr)}{\sqrt{(kr)}} A_n(\theta, \phi)$$

by § 3·32.

Now the equation of wave motions is unaffected if we multiply $x, y, z$ and $t$ by the same constant factor, i.e. if we multiply $r$ and $t$ by the same constant factor, leaving $\theta$ and $\phi$ unaltered; so that $A_n(\theta, \phi)$ may be taken to be independent† of the constant $k$ which multiplies $r$ and $t$.

Hence $\lim_{k \to 0} (k^{-n} V_n)$ is a solution of the equation of wave motions, that is to say, $r^n A_n(\theta, \phi)$ is a solution (independent of $t$) of the equation of wave motions, and is consequently a solution of Laplace's equation. Hence $A_n(\theta, \phi)$

* This follows from the fact that Laplace's operator is an invariant for changes of rectangular axes.

† This is otherwise obvious, because $S_n$ may be taken independent of $k$. 

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is a surface harmonic of degree $n$. If we assume it to be permissible to take
$A_n(\theta, \phi)$ to be any such harmonic, we obtain the result that

$$e^{\text{int} r} r^{n+1} (k r) P_n^m (\cos \theta) \frac{\cos m \phi}{\sin m \phi}$$

is a solution of the equation of wave motions*; and the motion represented by
this solution has frequency $k c$.

To justify the assumption that $A_n(\theta, \phi)$ may be any surface harmonic of degree $n$, we
construct the normal solution of the equation of wave motions

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = \frac{v^2}{c^2} \frac{\partial^2 V}{\partial t^2},$$

which has factors of the form $e^{\text{int} \cos m \phi}$. The factor which involves $\theta$ must then be of
the form $P_n^m (\cos \theta)$; and the factor which involves $r$ is annihilated by the operator

$$\frac{d}{dr} \left( r^2 \frac{d}{dr} \right) - n (n+1) + k^2 r^2,$$

so that if this factor is to be analytic at the origin it must be a multiple of $J_{n+\frac{1}{2}}(kr)/r$.

4.82. Theorems derived from solutions of the equations of Mathematical
Physics.

It is possible to prove (or, at any rate, to render probable) theorems con-
cerning Bessel functions by a comparison of various solutions of Laplace's
equation or of the equation of wave motions.

Thus, if we take the function

$$e^{kz} J_0 \left( k \sqrt{\rho^2 + a^2 - 2a \rho \cos \phi} \right),$$

by making a change of origin to the point $(a, 0, 0)$, we see that it is a solution
of Laplace's equation in cylindrical-polar coordinates. This solution has $e^{kz}$ as
a factor and it is analytic at all points of space. It is therefore natural to
expect it to be expansible in the form

$$e^{kz} \left[ A_0 J_0 (k \rho) + 2 \sum_{m=1}^{\infty} (A_m \cos m \rho + B_m \sin m \rho) J_m (k \rho) \right].$$

Assuming the possibility of this expansion, we observe that the function under
consideration is an even function of $\phi$, and so $B_m = 0$; and, from the symmetry
in $\rho$ and $a$, $A_m$ is of the form $c_m J_m (ka)$, where $c_m$ is independent of $\rho$ and $a$.

We thus get

$$J_0 \left( k \sqrt{\rho^2 + a^2 - 2a \rho \cos \phi} \right) = \sum_{m=0}^{\infty} c_m J_m (k \rho) J_m (ka) \cos m \phi.$$

If we expand both sides in powers of $\rho$, $a$ and $\cos \phi$, and compare the
coefficients of $(k^2 \rho a \cos \phi)^m$, we get

$$c_m = 1,$$

and so we are led to the expansion*

$$J_n(k \sqrt{(\rho^2 + a^2 - 2a \rho \cos \phi)}) = \sum_{m=0}^{\infty} c_m J_m(k \rho) J_m(k a) \cos m \phi,$$

of which a more formal proof will be given in §11.2.

Again, if we take $e^{ikr(x+zt)}$, which is a solution of the equation of wave motions, and which represents a wave moving in the direction of the axis of $z$ from $+\infty$ to $-\infty$ with frequency $kc$ and wave-length $2\pi/k$, we expect this expression to be expansible† in the form

$$\left(\frac{2\pi}{kr}\right)^{\frac{1}{2}} e^{ikr} \sum_{n=0}^{\infty} c_n i^n J_{n+\frac{1}{2}}(kr) P_n(\cos \theta),$$

where $c_n$ is a constant; so that

$$e^{ikr \cos \theta} = \left(\frac{2\pi}{kr}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} c_n i^n J_{n+\frac{1}{2}}(kr) P_n(\cos \theta).$$

If we compare the coefficients of $(kr \cos \theta)^n$ on each side, we find that

$$\frac{i^n}{n!} = \left(\frac{2\pi}{kr}\right)^{\frac{1}{2}} c_n \frac{i^n}{2^{n+\frac{1}{2}} \Gamma(n+\frac{1}{2})} \cdot \frac{(2n)!}{2^n \cdot (n!)^2},$$

and so $c_n = n + \frac{1}{2}$; we are thus led to the expansion‡

$$e^{ikr \cos \theta} = \left(\frac{2\pi}{kr}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} (n + \frac{1}{2}) i^n J_{n+\frac{1}{2}}(kr) P_n(\cos \theta),$$

of which a more formal proof will be given in §11.5.

4.83. Solutions of the wave equation in space of $p$ dimensions.

The analysis just explained has been extended by Hobson§ to the case of the equation

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \ldots + \frac{\partial^2 V}{\partial x_p^2} = \frac{1}{\sigma^2} \frac{\partial^2 V}{\partial t^2}.$$ 

A normal solution of this equation of frequency $kc$ which is expressible as a function of $r$ and $t$ only, where

$$r = \sqrt{(x_1^2 + x_2^2 + \ldots + x_p^2)},$$

must be annihilated by the operator

$$\frac{\partial^2}{\partial t^2} + \frac{p-1}{r} \frac{\partial}{\partial r} + k^2$$

and so such a solution, containing a time-factor $e^{ikt}$, must be of the form

$$e^{ikt} \sum_{n=0}^{\infty} c_n \frac{i^n}{n!} (kr)^{n-\frac{1}{2}}.$$  

* This is due to Neumann, *Theorie der Besselschen Functionen* (Leipzig, 1867), pp. 59—65.
† The tesselar harmonics do not occur because the function is symmetrical about the axis of $z$.
‡ This expansion is due to Bauer, *Journal für Math.* lvi. (1859), pp. 104, 106.
Hobson describes the quotient \( C_{(p-1)}^{0} (kr)/(kr)^{\frac{p-2}{2}} \) as a cylinder function of rank \( p \); such a function may be written in the form

\[
\mathcal{C} (kr \mid p).
\]

By using this notation combined with the concept of \( p \)-dimensional space, Hobson succeeded in proving a number of theorems for cylinder functions of integral order and of order equal to half an odd integer simultaneously.

As an example of such theorems we shall consider an expansion for

\[
J [k \sqrt{(r^2 + a^2 - 2ar \cos \phi)} \mid p],
\]

where it is convenient to regard \( \phi \) as being connected with \( x_p \) by the equation \( x_p = r \cos \phi \). This function multiplied by \( e^{ikt} \) is a solution of the wave equation, and when we write \( \rho = r \sin \phi \), it is expressible as a function of \( \rho, \phi, t \) and of no other coordinates.

Hence

\[
e^{ikt} J [k \sqrt{(r^2 + a^2 - 2ar \cos \phi)} \mid p]
\]

is annihilated by the operator

\[
\frac{\partial^2}{\partial \rho^2} + \frac{p - 2}{p} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial x_p^2} + k^2,
\]

that is to say, by the operator

\[
\frac{\partial^2}{\partial r^2} + \frac{p - 1}{r} \frac{\partial}{\partial r} + \frac{(p - 2)}{r^2 \sin \phi} \frac{\partial}{\partial \phi} + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2}{\partial \phi^2} + k^2.
\]

Now normal functions which are annihilated by this operator are of the form

\[
\frac{\mathcal{C}_{n+\frac{p-1}{2}}(kr)}{(kr)^{\frac{p-1}{2}}} P_n (\cos \phi \mid p),
\]

where \( P_n (\cos \phi \mid p) \) is the coefficient* of \( a^n \) in the expansion of

\[
(1 - 2a \cos \phi + a^2)^{1-\frac{p}{2}}.
\]

By the reasoning used in § 482, we infer that

\[
J [k \sqrt{(r^2 + a^2 - 2ar \cos \phi)} \mid p]
\]

\[
= \frac{1}{(ka)^{\frac{p-1}{2}} (kr)^{\frac{p-1}{2}}} \sum_{n=0}^{\infty} A_n J_{n+\frac{p-1}{2}}(ka) J_{n+\frac{p-1}{2}}(kr) P_n (\cos \phi \mid p).
\]

Now expand all the Bessel functions and equate the coefficients of \( (k^2 a^2 \cos \phi)^n \) on each side; we find that

\[
\frac{2^n}{2^{2n+1} n! \Gamma(n + \frac{3}{2} p)} \frac{A_n}{2^n \Gamma(n + \frac{1}{2} p - 1)} = \frac{2^n \Gamma(n + \frac{1}{2} p - 1)}{n! \Gamma(\frac{1}{2} p - 1)}
\]

so that \( A_n = 2^{\frac{p-1}{2}} (n + \frac{3}{2} p - 1) \Gamma(\frac{1}{2} p - 1) \).

* So that, in Gegenbauer’s notation,

\[
P_n (\cos \phi \mid p) \equiv C_{n-1}^{p-1}(\cos \phi).
\]