automorphic function, then $\{P, z\} P'^{-2}$ is a function automorphic for the same group.

But between two automorphic functions of the same group, there subsists an algebraical equation: hence there is an algebraical equation between P and $\{P, z\} P'^{-2}$, that is, $P(z)$, *an automorphic function of* z, *satisfies a differential equation of the third order, the degree of which is the integer representing the number of irreducible zeros of* P *and the coefficients of which, where they are not derivatives of* P, *are functions of* P *only and not of the independent variable.*

This equation can be differently regarded. Take

$$y_1 = P'^{\frac{1}{2}}, \qquad y_2 = zP'^{\frac{1}{2}};$$

then it is easy to prove that

$$\frac{1}{y_1}\frac{d^2y_1}{dP^2} = \frac{1}{y_2}\frac{d^2y_2}{dP^2} = \tfrac{1}{2}\frac{\{P, z\}}{P'^2}.$$

The last fraction has just been proved to be an automorphic function of z; and therefore it is rationally expressible in terms of P and any other general function, say Q, automorphic for the group. Then y_1 and y_2 are independent integrals of the equation

$$\frac{d^2y}{dP^2} = y\phi(P, Q),$$

where Q and P are connected by the algebraical equation

$$F(P, Q) = 0.$$

Conversely, the quotient of two independent integrals of the equation

$$\frac{d^2y}{dP^2} = y\phi(P, Q),$$

where Q and P are connected by the algebraical equation

$$F(P, Q) = 0,$$

can be taken as an argument of which P and Q are automorphic functions: the class of the equation $F = 0$ is the class of the infinite group of substitutions for which P and Q are automorphic*.

Ex. One of the simplest set of examples of automorphic functions is furnished by the class of homoperiodic functions (§ 116). Another set of such examples arises in the triangular functions, discussed in § 275; they are automorphic for an infinite group, and the triangles have a circle for their natural limit. A third set is furnished by the polyhedral functions (§§ 276—279).

As a last set of examples, we may consider the modular-functions which were obtained by a special method in § 303.

* Klein remarks (*Math. Ann.*, t. xix, p. 143, note 4) that the idea of uniform automorphic functions occurs in a posthumous fragment by Riemann (*Ges. Werke*, number xxv, pp. 413—416). It may also be pointed out that the association of such functions with the linear differential equation of the second order is indicated by Riemann.

First, we consider them in illustration of the algebraical relations between functions automorphic for the same group. It follows, from the construction of the group and the relation of c to w, that, in the division of the plane by the group with Uw and Vw for its fundamental substitutions, where

$$Uw = w + 2, \qquad Vw = \frac{w}{1-2w},$$

there is only a single point in each of the regions for which c has an assigned value; hence, regarding c as an automorphic function of w, the number κ (§ 310) is unity. If there be any other function C of w, automorphic for this group, then between C and c there is an algebraical relation of degree in C equal to the number κ for c, that is, of the first degree in C. Hence *every function automorphic for the group, whose fundamental substitutions are U and V, where*

$$Uw = w + 2, \qquad Vw = \frac{w}{1-2w},$$

is a rational algebraical function of c.

In the same way, it can be inferred that *every function automorphic for the group, whose fundamental substitutions are*

$$Uw = w + 2, \qquad Tw = -\frac{1}{w},$$

is a rational, algebraical, function of cc'; and that *every function automorphic for the group, whose fundamental substitutions are*

$$Sw = w + 1, \qquad Tw = -\frac{1}{w},$$

that is, automorphic for all substitutions of the form $\dfrac{aw+b}{cw+d}$, where a, b, c, d are real integers, such that $ad - bc = 1$, is a rational algebraical function of $J = \dfrac{(c^2 - c + 1)^3}{c^2(c-1)^2}$.

Secondly, in illustration of the general theorem relating to the differential equation of the third order which is characteristic of an automorphic function, we consider the quantity c as a function of the quotient of the quarter-periods. Let z denote $\dfrac{iK'}{K}$: then because every function automorphic for the same group of substitutions as c is a rational function of c, we have

$$\frac{\{c, z\}}{c'^2} = \text{rational function of } c;$$

and therefore, by a property of the Schwarzian derivative,

$$\{z, c\} = -\text{ same rational function of } c.$$

By known formulæ of elliptic functions, it is easy to shew that

$$\{z, c\} = \frac{1 - c + c^2}{2c^2(1-c)^2},$$

thus verifying the general result.

Similarly, it follows that $\left\{\dfrac{iK'}{K}, \theta\right\}$, where $\theta = cc'$, is a rational function of cc', the actual value being given by

$$\left\{\frac{iK'}{K}, \theta\right\} = \frac{1 - 5\theta + 16\theta^2}{2\theta^2(1-4\theta)^2};$$

and that $\left\{\dfrac{iK'}{K}, J\right\}$ is a rational function of J, the actual value being given by

$$\left\{\frac{iK'}{K}, J\right\} = \frac{16J^2 - 123J - 330}{2J^2(4J-27)^2}.$$

In this connection a memoir by Hurwitz* may be consulted.

* *Math. Ann.*, t. xxxiii, (1889), pp. 345—352.

The preceding application to differential equations is only one instance in the general theory which connects automorphic functions with linear differential equations having algebraical coefficients. This development belongs to the theory of differential equations rather than to the general theory of functions: its exposition must be reserved for another place.

Here my present task comes to an end. The range of the theory of functions is vast, its ramifications are many, its development seems illimitable: an idea of its freshness and its magnitude can be acquired by noting the results, and appreciating the suggestions, contained in the memoirs of the mathematicians who are quoted in the preceding pages.

GLOSSARY

OF TECHNICAL TERMS USED IN THE THEORY OF FUNCTIONS.

(*The numbers refer to the pages, where the term occurs for the first time in the book or is defined.*)

Abbildung, conforme, 11.
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Algebraical addition-theorem, 297.
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Algebraical function determined by an equation, 161.
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Analytical curve, 409, 423, 530.
Analytic function, monogenic, 56.
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Arithmetic mean, method of the, 408.
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Betrag, absoluter, 3.
Bien défini, 161.
Bifacial surface, 325.
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GENERAL INDEX.

(The numbers refer to the pages.)