so that $$\arg.\left(w-\frac{a}{c}\right)+\arg.\left(z+\frac{d}{c}\right)=\pi.$$

This at once shews that, whatever be the value of $\frac{a}{c}$ and of $\frac{d}{c}$, the points A, E are homologous, and likewise the points B, D. Hence to obtain a corner homologous to a given corner we start from the corner, describe the edge of the polygon beginning there, then describe in the same direction* the conjugate edge: the extremity of that edge is a homologous corner.

The process may now be reapplied, beginning with the last point; and it can be continued, each stage adding one point to the cycle, until we either return to the initial point or until we are met by an edge of the second kind. In the former case we have a completed cycle, which may be regarded as a *closed* cycle. In the latter case we can proceed no further, as edges of the second kind are not ranged in conjugate pairs; but, resuming at the initial point we apply the process with a description in the reverse direction until we again arrive at an edge of the second kind: again we have a cycle, which may be regarded as an *open* cycle.

In the case of a closed cycle, if one of the included points be of the first category, then all the points are of the first category: the cycle itself is then said to be of the first category. If one of the points be of the second category, then since no edge of the second kind is met in the description, all the edges met are of the first kind; and therefore all the points, lying on the axis of x and being the intersections of edges of the first kind, are of the second category: the cycle itself is then said to be of the second category.

Open cycles will contain points of the third category: they may also contain points of the second category for points both of the second and of the third categories lie on the axis of x, and homology of the points does not imply conjugacy of all edges of which they are extremities. Such cycles are said to be of the third category.

It thus appears that the cycles can be derived when the arrangement in conjugate pairs of edges of the first kind is given; and it is easy to see that the number of open cycles is equal to the number of edges of the second kind.

We may take one or two examples. For a quadrilateral, in which the conjugate pairs are 1, 4; 2, 3—the numbers being as in the figure—we have by the above process A, AB, DA, A: that is, A is a cycle by itself. Then B, BC, CD, D, DA, AB, B: that is, B and D form a cycle; and then C, CD, BC, C, that is, C is a cycle by itself. The cycles are therefore three, namely, A; B, D; C.

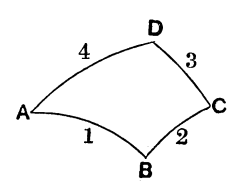

Fig. 112.

* This is necessary: the direction is easily settled for a complete polygon the sides of which are described in positive or in negative direction throughout.

For a hexagon, in which the conjugate pairs are 1, 5; 2, 4; 3, 6, the cycles are two, namely, A, F, D, C and B, E. If the conjugate pairs be

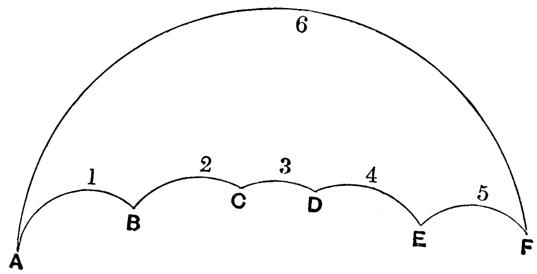

Fig. 113.

1, 6; 2, 5; 3, 4, the cycles are four, namely, A; B, F; C, E; D. If the conjugate pairs be 1, 4; 2, 5; 3, 6 the cycles are two, namely, A, C, E; B, D, F.

For a pentagon, with one edge of the second kind as in the figure and

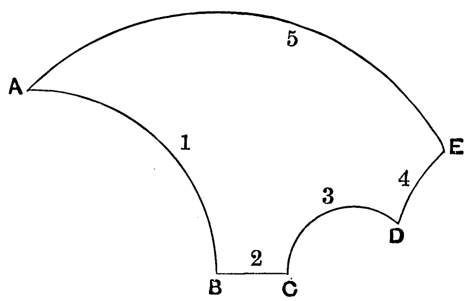

Fig. 114.

having 1, 3; 4, 5 as the conjugate pairs, the cycles are three, namely, E; A, D; B, C; the last being open and of the third category.

For a quadrilateral as in the figure, having three corners on the axis of x and 1, 2; 3, 4 as the arrangement of its conjugate pairs, the cycles are D; A, C; B; the last two being of the second category.

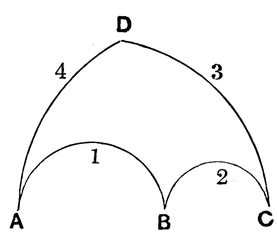

Fig. 115.

We have now to consider the angles of the polygons taken internally. It is evident that at any corner of the second category, the angle is zero, for it is the angle between two circles meeting on their line of centres; and that at any corner of the third category the angle is right. There therefore remain only the angles at corners of the first category. Let $A_1, A_2, \ldots, A_n$ be the corners in a cycle of the first category and denote the angles by the same letters.

Since A_1 and A_2 are homologous corners, they are extremities of conjugate edges. Apply to the plane, in the vicinity of A_2, the substitution which changes the edge ending in A_2 to its conjugate ending in A_1: then the point A_2 is transferred to the point A_1; one edge at A_2 coincides with its conjugate at A_1 and the other edge at A_2 makes an angle A_2 with it, because of the substitution which conserves angles. The latter edge was the edge which followed A_2 in the cycle for the derivation of A_3: we take its conjugate ending in A_3, and treat these and the points A_2 and A_3 as before for A_1 and A_2 and their conjugate edges, namely, by using the substitutions transforming conjugate edges and passing from A_3 to A_2 and then those from A_2 to A_1.

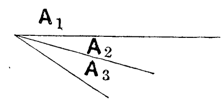

Fig. 116.

Proceeding in this way round the cycle, we shall have

(1) a series of lines at the point, each line between two angles being one of the conjugate edges on which the two corners lie:

(2) the angles corresponding to the corners taken in cyclical order.

Hence after n such operations we shall again reach an angle A_1. If the edge do not coincide with the first edge, we repeat the set of n operations: and so on.

Now all these substitutions lead to the construction of the various regions meeting in A, which are to occupy all the plane round A, and no two of which are to contain a point which does not lie on an edge. Hence after the completion of some set of operations, say the pth set, the edges of A_1 will coincide with their edges of the first angle A_1; and therefore

$$p(A_1 + A_2 + \ldots + A_n) = 2\pi,$$

so that

$$A_1 + A_2 + \ldots + A_n = \frac{2\pi}{p}.$$

Hence *the sum of the angles at the corners, in a cycle of the first category, is a submultiple of* 2π.

Further, if q be the number of polygons at A, we have

$$np = q.$$

COROLLARY 1. *For a cycle of the second category—it is a closed cycle—both p and q are infinite.*

The cycle contains only a finite number of corners, because the polygon has only a finite number* of edges: as each corner is of the second category,

* If the number be infinite, the edges must be infinitesimal in length unless the perimeter of each of the polygons is infinite: each of these alternatives is excluded.

The reason for finiteness (§ 282) in the number of fundamental substitutions in the group is now obvious: their number is one-half of the number of edges of the first kind.

the angle is zero: and therefore the repetition of the set of operations can be effected without limit. Hence p is infinite; and, as n polygons at a corner are given by each set of operations, the number q of polygons is infinite.

COROLLARY 2. *Corresponding to every cycle of the first category, there is a relation among the fundamental substitutions of the group.*

Let f_{12} be the substitution interchanging the conjugate edges through A_1 and A_2; f_{23} the substitution interchanging the conjugate edges through A_2 and A_3; and so on. Let U denote

$$f_{12}^{-1} \cdot f_{23}^{-1} \cdot f_{34}^{-1} \dots \cdot f_{n-1,n}^{-1}(z);$$

then

$$U^p(z) = z.$$

For U is the substitution which reproduces the polygon with the angle A_1 at A_1; and this substitution is easily seen, after the preceding explanation, to be periodic of order p. Moreover, this substitution U is elliptic.

288. The following characteristics of the fundamental region have now been obtained:

(i) It is a convex polygon, the edges of which are either arcs of circles with their centres on the axis of x or are portions of the axis of x:

(ii) The edges of the former kind are even in number and can be arranged in conjugate pairs: there is a substitution for which the edges of a conjugate pair are congruent; if this substitution change one edge a of the pair into a', it changes the given region into the region on the other side of a':

(iii) The corners of the polygon can be arranged in cycles of one or other of three categories:

(iv) The angles at corners in a cycle of the second category are zero: each of the angles at corners in a cycle of the third category is right: the sum of the angles at corners in a cycle of the first category is a submultiple of 2π.

Let there be an infinite discontinuous group of substitutions, such that its fundamental substitutions are characterised by the occurrence of the foregoing properties in the edges and the angles of the geometrically associated region: and let the whole group of substitutions be applied to the region.

Then the half-plane on the positive side of the axis of x is covered: no part is covered more than once, and no part is unassigned to regions. It is easy to see in a general way how this given condition is satisfied by the various properties of the regions. Since the edges of the first kind in the initial region can be arranged in conjugate pairs, it is so with those edges in every region: and the substitution, which makes them congruent, makes one of them to coincide with the homologue of the other for the neighbouring region, so that no part is unassigned. No part is covered twice, for the initial region is a normal convex polygon and therefore every region is a normal convex polygon: the edges are homologous from region to region, and form a common boundary. The angle of intersection with a given arc is sufficient to fix the edge of the consecutive polygon: for an arc of a circle, making on one side an assigned angle with a given arc and having its centre on the axis, is unique. At every corner of any polygon, there will be a number of polygons: the corners which coincide there are, for the different polygons, the corners homologous with a cycle in the original region: and the angles belonging to those corners fill up, either alone or after an exact number of repetitions, the full angle round the point.

We have seen that the substitution, which passes from a polygon at a point to the same polygon, after n polygons, reproduces the angular point at the same time as it reproduces the polygon; the point is a fixed point of an elliptic substitution. Similarly, if the point belong to a cycle of the second category, n is infinite and the substitution does not change the point, which is therefore a fixed point of the substitution; as the fixed point is on the axis, the substitution is parabolic (§ 292).

The preceding are the essential properties of the regions, which are sufficient for the division of the half-plane when a group is given, and therefore by reflexion through the axis of x, they are sufficient for the division of the other half-plane.

The position of corners of the first category, and the orientation of edges meeting in those corners, are determinate when the group is supposed given: within certain limits, half of the corners of the third category can be arbitrarily chosen.

289. In the preceding investigation, the group has been supposed given: the problem was the appropriate division of the plane. The converse problem occurs when a fundamental region, with properties appropriate for the division of the half-plane, is given: it is the determination of the group. The fundamental substitutions of the group are those which transform an edge into its conjugate, and they are to be real—conditions which, by § 258, are sufficient for their construction. The whole group of substitutions is obtained by combining those that are fundamental. The complete division of the half-plane is effected, by applying to each polygon in succession the series of fundamental substitutions and of their first inverses.

It is evident that a given division of the plane into regions determines the group uniquely: but, as has already been seen in the general explanation, the existence of a group with the requisite properties does not imply a unique division of the plane.

As an example, let the fundamental substitutions be required when a quadrilateral as in Fig. 112, having 1, 2; 3, 4 for the conjugate pairs of edges, is given as a fundamental region. The cycles of the corners are B; D; A, C; so that

$$B=\frac{2\pi}{l}, \quad D=\frac{2\pi}{m}, \quad A+C=\frac{2\pi}{n},$$

where l, m, n are integers.

The simplest case has already been treated, § 284: there, $l=2$, $m=\infty$, $n=3$, $A=C$; the region is a triangle, really a quadrilateral with two edges as conterminous arcs of the same circle. We shall therefore suppose this case excluded; we take the case next in point of simplicity, viz. $l=2$, $A=C$. Then AB and BC are conterminous arcs of one circle: we shall take the centre of this circle to be the origin, its radius unity and B on the axis of y; then B is a fixed point of the substitution, which changes AB into BC. The substitution is

$$w=-\frac{1}{z};$$

it is one of the two fundamental substitutions.

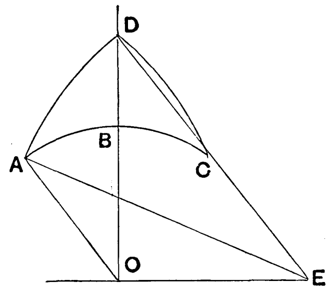

Fig. 117.

Evidently $A=\frac{\pi}{n}$, $ADB=\frac{\pi}{m}$. Let E be the centre of the circle AD, and ρ its radius: then $OAE=\frac{\pi}{n}$, $ODE=\frac{\pi}{2}-\frac{\pi}{m}$, and so

$$\rho^2+1-2\rho\cos\frac{\pi}{n}=OE^2=\rho^2\cos^2\frac{\pi}{m},$$

whence

$$\rho\sin^2\frac{\pi}{m}=\cos\frac{\pi}{n}+\left(\cos^2\frac{\pi}{n}-\sin^2\frac{\pi}{m}\right)^{\frac{1}{2}},$$

the negative sign of the radical corresponding to the case when D lies below ABC. The radius ρ must be real and therefore

$$\frac{1}{n}+\frac{1}{m}\leqslant\tfrac{1}{2}:$$

we omit the case of $m=\infty$, and therefore $n>2$.

The fundamental substitution, which changes AD into CD, has D and the complex conjugate to D for its fixed points: these points are $\pm i\rho\sin\frac{\pi}{m}$. The argument of the multiplier is $\frac{2\pi}{m}$, being the angle ADC: hence the substitution is

$$\frac{w-i\rho\sin\frac{\pi}{m}}{w+i\rho\sin\frac{\pi}{m}}=\frac{z-i\rho\sin\frac{\pi}{m}}{z+i\rho\sin\frac{\pi}{m}}e^{\frac{2\pi i}{m}},$$

which reduces to

$$w=\frac{z\cos\frac{\pi}{m}+\rho\sin^2\frac{\pi}{m}}{-\frac{z}{\rho}+\cos\frac{\pi}{m}},$$

where ρ has the value given by the above equation.

This substitution, and the substitution $w=-\frac{1}{z}$, are the fundamental substitutions of the group. The special illustration in § 284 gives

$$m=\infty\,,\ \rho=\infty\,,\ n=3,\ \rho\sin^2\frac{\pi}{m}=2\cos\frac{\pi}{n}=1\;;$$

the special form therefore is

$$w=z+1.$$

Taking $\cos\frac{\pi}{m}=a$, $\cos\frac{\pi}{n}=b$, $\Delta=(a^2+b^2-1)^{\frac{1}{2}}$, we have $\rho\,(1-a^2)=b+\Delta$; the second fundamental substitution is

$$w=Sz=\frac{az+\Delta+b}{(\Delta-b)\,z+a}.$$

It is easy to see that

$$T^2=1,\ S^m=1,\ (TS)^n=1,$$

where $Tz=-\frac{1}{z}$; the complete figure can be constructed as in § 284.

An interesting figure occurs for $m=4$, $n=6$.

In the same way it may be proved that, if an elliptic substitution have $re^{\pm\theta i}$ for its common points and 2Θ for the argument of its multiplier, its expression is

$$w=\frac{Az+B}{Cz+D},$$

where $\qquad A=\frac{\sin(\theta-\Theta)}{\sin\theta},\qquad B=r\frac{\sin\Theta}{\sin\theta},\qquad C=-\frac{1}{r}\frac{\sin\Theta}{\sin\theta},\qquad D=\frac{\sin(\theta+\Theta)}{\sin\theta}.$

Taking now the more general case where $B=\frac{2\pi}{l}$, $D=\frac{2\pi}{m}$, $A+C=\frac{2\pi}{n}$, let B (in figure 112) be the point $be^{\beta i}$, and A the point $ae^{\alpha i}$. Then the substitution which transforms AB into BC is the above, when $\theta=\beta$, $r=b$, $\Theta=B$, so that, if C be $ce^{\gamma i}$,

$$ce^{\gamma i}=\frac{a\sin(\beta-B)\,e^{\alpha i}+b\sin B}{-\frac{a}{b}\sin Be^{\alpha i}+\sin(\beta+B)},$$

giving two relations among the constants.

Similarly two more relations will arise out of the substitution which transforms CD into DA. And three relations are given by the conditions that the sum of the angles at A and C is an aliquot part of 2π, and that each of the angles B and D is an aliquot part of 2π.

290. All the substitutions hitherto considered have been real: we now pass to the consideration of those which have complex coefficients. Let

$$\frac{\alpha z+\beta}{\gamma z+\delta}$$

be such an one, supposed discontinuous: then the effect on a point is obtained by displacing the origin, inverting with respect to the new position, reflecting through a line inclined to the axis of x at some angle, and again displacing the origin. The displacements of the origins do not alter the character of relations of points, lines and curves: so that the essential parts of the transformation are an inversion and a reflexion.

Let a group of real substitutions of the character considered in the preceding sections be transformed by the foregoing single complex substitution: a new group

$$\left(\frac{\alpha z+\beta}{\gamma z+\delta},\ \frac{a\dfrac{\alpha z+\beta}{\gamma z+\delta}+b}{c\dfrac{\alpha z+\beta}{\gamma z+\delta}+d}\right)$$

will thus be derived. The geometrical representation is obtained through transforming the old geometrical representation by the substitution

$$\left(\frac{\alpha z+\beta}{\gamma z+\delta},\ z\right),$$

so that the new group is discontinuous.

The original group left the axis of x unchanged, that is, the line $z=z_0$ was unchanged; hence the substitutions

$$\left(\frac{\alpha z+\beta}{\gamma z+\delta},\ \frac{a\dfrac{\alpha z+\beta}{\gamma z+\delta}+b}{c\dfrac{\alpha z+\beta}{\gamma z+\delta}+d}\right)$$

will leave unchanged the line which is congruent with $z=z_0$ by the substitution $\left(\dfrac{\alpha z+\beta}{\gamma z+\delta},\ z\right)$. This line is

$$\frac{-\delta z+\beta}{\gamma z-\alpha}=\frac{-\delta_0 z_0+\beta_0}{\gamma_0 z_0-\alpha_0},$$

or it may be taken in the form

$$\text{imaginary part of } \frac{-\delta z+\beta}{\gamma z-\alpha}=0.$$

It is a circle, being the inverse of a line; it is unaltered by the substitutions of the new group, and it is therefore called* the *fundamental circle* of this group. The group is still called Fuchsian (p. 606, note).

The half-planes on the two sides of the axis of x are transformed into the two parts of the plane which lie within and without the fundamental circle respectively: let the positive half-plane be transformed into the part within the circle.

With the group of real substitutions, points lying above the axis of x are transformed into points also lying above the axis of x, and points below into points below: hence with the new group, points within the fundamental circle are transformed into points also within the circle, and points without into points without.

* Klein uses the word *Hauptkreis*.

The division of the half-plane into curvilinear polygons is changed into a division of the part within the circle into curvilinear polygons. The sides of the polygons either are circles having their centres on the axis of x, that is, cutting the axis orthogonally, or they are parts of the axis of x: hence the sides of the polygons in the division of the circle either are arcs of circles cutting the fundamental circle orthogonally or they are arcs of the fundamental circle.

The division of the part of the plane without the circle is the transformation of the half-plane below the axis of x, which is a mere reflexion in the axis of x of the half-plane above: thus the division is characterised by the same properties as characterise the division of the part within the fundamental circle. But when the division of the part within the circle is given, the actual division of the part without it can be more easily obtained by inversion with the centre of the fundamental circle as centre and its radius as radius of inversion.

This process is justified by the proposition that conjugate complexes are transformed by the substitution $\left(\dfrac{\alpha z+\beta}{\gamma z+\delta},\ z\right)$ into points which are the inverses of one another with regard to the fundamental circle. For a system of circles can be drawn through two conjugate complexes, cutting the real axis orthogonally: when the transformation is applied, we have a system of circles, orthogonal to the fundamental circle and passing through the two corresponding points. The latter are therefore inverses with regard to the fundamental circle.

This proposition can also be proved in the following elementary manner.

Let OC, the axis of x, be inverted, with A as the centre of inversion, into a circle: P and Q be two conjugate complexes, and let AP cut axis of x in C: let CQ cut the diameter of the circle in R. Since OC bisects PQ, it bisects AR; and therefore the centre of the circle is the inverse of R.

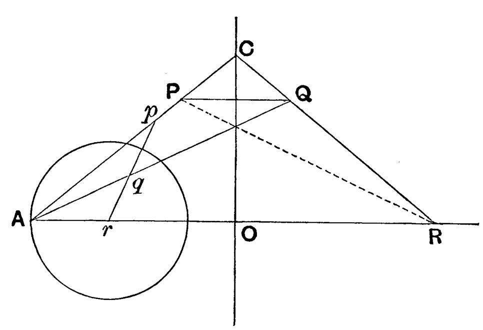

Fig. 118.

Let p and q be the inverses of P and Q: join pq, qr. Then the angle $pqQ = CPQ = CQP$, and $Aqr = CRO$: thus pqr is a straight line. Also

$$\frac{qr}{Aq} = \frac{QR}{AR} = \frac{AP}{AR} = \frac{Ar}{Ap},$$

and
$$\frac{pr}{Ap} = \frac{PR}{AR} = \frac{AQ}{AR} = \frac{Ar}{Aq},$$

so that
$$rp \,.\, rq = Ar^2.$$

Thus p and q are inverses of each other, relative to r and with the radius of the fundamental circle as radius. Transference of origin and reflexion in a straight line do not alter these properties: and therefore p and q, the transformations of the conjugate P and Q, are inverses of one another with regard to the fundamental circle.

Hence with the present group, constructed from an infinite discontinuous group of real substitutions transformed by a single complex substitution, the fundamental circle has the same importance as the axis of real quantities in the group of real substitutions. It is of finite radius, which will be taken to be unity: its centre will be taken to be the origin. The area within it is divided into regions congruent with one another by the substitutions of the group: the whole of the area is covered by the polygons, but no part is covered more than once.

All the points, homologous with a given point z within the circle, lie within the circle: each polygon contains only one of such a set of homologous points.

The angular points of a polygon can be arranged in cycles which are of three categories. The sum of the angles at points in a cycle of the first category is unchanged by the substitution; it is equal to an aliquot part of 2π. At points in a cycle of the second category each angle is zero: at points in a cycle of the third category each angle is right.

In fact, all the properties obtained for the division of the plane into polygons now hold for the division of the circle into polygons associated with the group

$$\left(\frac{\alpha z+\beta}{\gamma z+\delta},\ \frac{a\dfrac{\alpha z+\beta}{\gamma z+\delta}+b}{c\dfrac{\alpha z+\beta}{\gamma z+\delta}+d}\right),$$

provided we make the changes that are consequent on the transformation of the axis of x into the fundamental circle.

The form of the substitution

$$w=\frac{\alpha z+\beta}{\gamma z+\delta},$$

which secures that the fundamental circle in the w-plane shall be of radius unity and centre the origin, is easily obtained.

It has been proved that inverse points with respect to the circle correspond to conjugate complexes; hence $w=0$ and $w=\infty$ correspond to two conjugate complexes, say λ and λ_0, and therefore

$$w=\kappa\frac{z-\lambda}{z-\lambda_0},$$

where $|\kappa|=1$ because the radius of the fundamental circle is to be unity. The presence of this factor κ is equivalent to a rotation of the w-plane about the origin. As the origin is the centre of the fundamental circle, the circle is unaltered by such a change: and therefore, without affecting the generality of the substitution, we may take $\kappa=1$, so that now

$$w=\frac{z-\lambda}{z-\lambda_0},$$

where λ is an arbitrary complex constant. The substitution is not in its canonical form, which however can at once be deduced.

291. It has been seen, in § 260, that, when any real substitution is parabolic or hyperbolic, then practically an infinite number of points coincide with the fixed point of the substitution when it is repeated indefinitely, whatever be the point z initially subjected to the transformation; this fixed point lies on the axis of x, and is called an essential singularity of the substitution. When we consider such points in reference to automorphic functions, which are such as to resume their value when their argument is subjected to the linear substitutions of the group, then at such a point the function resumes the value which it had at the point initially transformed; that is, in the immediate vicinity of such a fixed point of the substitution, the function acquires any number of different values: such a point is an essential singularity of the function. Hence the essential singularities of the group are the essential singularities of the corresponding function.

Now all the essential singularities of a discontinuous group lie on the axis of x when the group is real; the line may be or may not be a continuous line of essential singularity. If, for example, x be any such point for the group of §§ 283, 284 which is characteristic of elliptic modular-functions, then all the others for that group are given by

$$\frac{ax+b}{cx+d},$$

where a, b, c, d are integers, subject to the condition $ad-bc=1$: and therefore all the essential singularities are given by rational linear transformations. For points on the real axis, this group is improperly discontinuous: and therefore for this group the axis of x is a continuous line of essential singularity.

Hence when we use the transformation $\left(\frac{\alpha z+\beta}{\gamma z+\delta}, z\right)$ to deduce the division of the fundamental circle into regions, the essential singularities of the new group are points on the circumference of the fundamental circle: the circumference is or is not a continuous line of essential singularity for the function or the group*, according as the group is properly or improperly discontinuous for the circle.

292. It is convenient to divide the groups into families, the discrimination adopted by Poincaré being made according to the categories of cycles of angular points in the polygons into which the group divides the plane. The group is of the

1st family, if the polygon have cycles of the	1st	category only,
2nd	2nd	...............,
3rd	3rd	...............,
4th	2nd and 3rd	...............,

* Poincaré calls the group Fuchsian, both when all the coefficients are real and when they arise from the transformation of such an infinite group by a single substitution that has imaginary coefficients. A convenient resumé of his results is given by him in a paper, *Math. Ann.*, t. xix, (1882), pp. 553—564.

5th family, if the polygon have cycles of the 1st and 3rd categories only,
6th .... 1st and 2nd ....,
7th .... all three categories.

Thus in the polygons, associated with groups of the 1st, the 2nd, and the 6th families, all the edges are of the first kind; in the polygons associated with groups of the remaining families, edges of the second kind occur.

A subdivision of some of the families is possible. It has been proved that the sum of the angles in a cycle of the first category is a submultiple of 2π. If the sum is actually 2π, the cycle is said to belong to the first sub-category: if it be less than 2π (being necessarily a submultiple), the cycle is said to belong to the second sub-category. And then, if all the cycles of the polygon belong to the first sub-category, the group is said to belong to the first order in the first family: if the polygon have any cycle belonging to the second sub-category, the group is said to belong to the second order in the first family.

It has been proved in § 288 that a corner belonging to a cycle of the second category is not changed by the substitution which gives the contiguous polygons in succession; the corner is a fixed point of the substitution, so that the substitution is either parabolic or hyperbolic. In his arrangement of families, Poincaré divided the cycles of the second category into cycles of two sub-categories, according as the substitution is parabolic or hyperbolic: but Klein has proved* that there are no cycles for hyperbolic substitutions, and therefore the division is unnecessary. The families of groups, the polygons associated with which have cycles of the second category, are the second, the fourth, the fifth and the seventh.

There is one very marked difference between the set of families, consisting of the first, the second and the sixth, and the set constituted by the remainder.

No polygon associated with a real group in the former set has an edge of the second kind: and therefore the only points on the axis taken account of in the division of the plane are the essential singularities of the group. The domain of any ordinary point on the axis in the vicinity of each of the essential singularities is infinitesimal: and therefore the axis of x is taken account of in the division of the plane only in so far as it contains essential singularities of the group and the functions. This, of course, applies equally to the transformed configuration in which the conserved line is the fundamental circle: and therefore, in the division of the area of the circle, its circumference is taken account of only in so far as it contains essential singularities of the groups and the functions.

But each polygon associated with a real group in the second set of families has an edge of the second kind: the groups still have all their

* *Math. Ann.*, t. xl, (1892), p. 132.

essential singularities on the axis of x (or on the fundamental circle) and at least some of them are isolated points; so that the domain of an ordinary point on the axis is not infinitesimal. Hence parts of the axis of x (or of the circumference of the fundamental circle) fall into the division of the bounded space.

293. There is a method of ranging groups which is of importance in connection with the automorphic functions determined by them.

The upper half of the plane of representation has been divided into curvilinear polygons; it is evident that the reflexion of the division, in the axis of real quantities, is the division of the lower half of the plane. Let the polygon of reference in the upper half be R_0 and in the lower half be R_0', obtained from R_0 by reflexion in the axis of real quantities. Then, if the group belong to the set, which includes the first, the second and the sixth families, R_0 and R_0' do not meet except at those isolated points, which are polygonal corners of the second category. But if the group belong to the set which includes the remaining families, then R_0 and R_0' are contiguous along all edges of the second kind, and they may be contiguous also at isolated points as before.

In the former case R_0 and R_0' may be regarded as distinct spaces, each fundamental for its own half-plane. Let R_0 have $2n$ edges which can be arranged in n conjugate pairs, and let q be the number of cycles all of which are closed; each point in one edge corresponds to a single point in the conjugate edge. Let the surface included by the polygon R_0 be deformed and stretched in such a manner that conjugate edges are made to coincide by the coincidence of corresponding points. A closed surface is obtained. For each pair of edges in the polygon there is a line on the surface, and for each cycle in the polygon there is a point on the surface in which lines meet; and the lines make up a single curvilinear polygon occupying the whole surface. The process is reversible; and therefore the connectivity of the surface is an integer which may properly be associated with the fundamental polygon.

When two consecutive edges are conjugate, their common corner is a cycle by itself. The line, made up of these two edges after the deformation, ends in the common corner which has become an isolated point; this line can be obliterated without changing the connectivity. The obliteration annuls two edges and one cycle of the original polygon: that is, it diminishes n by unity and q by unity. Let there be r such pairs of consecutive edges. The deformed surface is now occupied by a single polygon, with $n-r$ sides and $q-r$ angular points; so that, if its connectivity be $2N+1$, we have (§ 165)

$$\begin{aligned}2N &= 2+(n-r)-1-(q-r)\\ &= n+1-q.\end{aligned}$$

The group may be said to be of *class N*.

In the latter case, the combination of R_0 and R_0' may be regarded as a single region, fundamental for the whole plane. Let R_0 have $2n$ edges of the first kind and m of the second kind, and let q be the number of closed cycles: the number of open cycles is m. Then R_0' has $2n$ edges of the first kind and q closed cycles; it has, in common with R_0, the m edges of the second kind and the m open cycles. The correspondence of points on the edges of the first kind is as before. Let the surface included by R_0 and R_0' taken together be deformed and stretched in such a manner that conjugate edges coincide by the coincidence of corresponding points on those edges. A closed surface is obtained. As the process is reversible, the connectivity of the surface thus obtained is an integer which may properly be associated with the fundamental polygon.

This integer is determined as before. For each pair of edges of the first kind in either polygon, a line is obtained on the surface; so that $2n$ lines are thus obtained, n from R_0 and n from R_0'. Each of the common edges of the second kind is a line on the surface, so that m lines are thus obtained. The total number of lines is therefore $2n+m$. For each of the closed cycles there is a point on the surface in which lines, obtained through the deformation of edges of the first kind, meet: their number is $2q$, each of the polygons providing q of them. For each of the open cycles there is a point on the surface in which one of the m lines divides one of the n lines arising through R_0 from the corresponding line arising through R_0': the number of these points is m. The total number of points is therefore $2q+m$.

The total number of polygons on the surface is 2. Hence, if the connectivity be $2N+1$, we have (§ 165)

$$\begin{aligned} 2N &= 2 + 2n + m - (2q + m) - 2 \\ &= 2n - 2q. \end{aligned}$$

The group may be said to be of *class* N.

Thus for the generating quadrilateral in figure 112 (p. 596), the class of the group is zero when the arrangement of the conjugate pairs is 1, 2; 3, 4: and it is unity when the arrangement of the pairs is 1, 3; 2, 4. For the generating hexagon in figure 113 (p. 597), the class of the group is zero when the arrangement of the conjugate pairs is 1, 6; 2, 5; 3, 4: and it is unity when the arrangement of the pairs is 1, 4; 2, 5; 3, 6. For the generating pentagon in figure 114 (p. 597), the class of the group is zero when the arrangement of the conjugate pairs is 1, 3; 4, 5: and it is two, when the arrangement of the pairs is 1, 4; 3, 5. For a generating polygon, bounded by $2n$ semi-circles each without all the others and by the portions of the axis of x, the number of closed cycles is zero: hence $N=n$.

294. In all the groups, which lead to a division of a half-plane or of a circle into polygons, the substitutions have real coefficients or are composed of real substitutions and a single substitution with complex coefficients: and thus the variation in the complex part of the coefficients in the group is strictly limited. We now proceed to consider groups of substitutions

$$\left(z, \frac{\alpha z+\beta}{\gamma z+\delta}\right),$$

in which the coefficients are complex in the most general manner: such groups, when properly discontinuous, are called *Kleinian*, by Poincaré.

The Fuchsian groups conserve a line, the axis of x, or a circle, the fundamental circle: the Kleinian groups do not conserve such a line or circle, common to the group. Every substitution can be resolved into two displacements of origin, an inversion and a reflexion, as in § 258. The inversion has for its centre the point $-\delta/\gamma$, being the origin after the first displacement; the reflexion is in the line through this point making with the real axis an angle $\pi - 2 \arg. \gamma$. The only line left unaltered by these processes is one which makes an angle $\frac{1}{2}\pi - \arg. \gamma$ with the real axis and passes through the point; and the final displacement to the point α/γ will in general displace this line. Moreover, arg. γ is not the same for all substitutions; there is therefore no straight line thus conserved common to the group.

Similar considerations shew that there is no fundamental circle for the group, persisting untransformed through all the substitutions.

Hence the Kleinian groups conserve no fundamental line and no fundamental circle: when they are used to divide the plane, the result cannot be similar to that secured by the Fuchsian groups. As will now be proved, they can be used to give relations between positions in space, as well as relations between positions merely in a plane.

The lineo-linear relation between two complex variables, expressed as a linear substitution, has been proved (§ 261) to be the algebraical equivalent of any even number of inversions with regard to circles in the plane of the variables: this analytical relation, when developed in its geometrical aspect, can be made subservient to the correlation of points in space.

Let spheres be constructed which have, as their equatorial circles, the circles in the system of inversions just indicated; let inversions be now carried out with regard to these spheres, instead of merely with regard to their equatorial circles. It is evident that the consequent relations between points in the plane of the variable z are the same as when inversion is carried out with regard to the circles: but now there is a unique transformation of points that do not lie in the plane. Moreover, the transformation possesses the character of conformal representation, for it conserves angles and it secures the similarity of infinitesimal figures: points lying above the plane of z

invert into points lying above the plane of z, so that the plane of z is common to all these spherical inversions and therefore common to the substitutions, the analytical expression of which is to be associated with the geometrical operation; and a sphere, having its centre in the plane of the complex z is transformed into another sphere, having its centre in that plane, so that the equatorial circles correspond to one another.

Through any point P in space, let an arbitrary sphere be drawn, having its centre in the plane of the complex variable, say, that of the coordinates ξ, η. It will be transformed, by the various inversions indicated, into another sphere, having its centre also in the plane of ξ, η and passing through the point Q obtained from P as the result of all the inversions; and the equatorial planes will correspond to one another.

Let the sphere through Q be

$$(\xi' - a)^2 + (\eta' - b)^2 + \zeta'^2 = r'^2,$$

or

$$\xi'^2 + \eta'^2 + \zeta'^2 - 2a\xi' - 2b\eta' + k = 0.$$

Hence, if Q be determined by

$$z' = \xi' + i\eta', \quad z_0' = \xi' - i\eta', \quad \rho'^2 = \xi'^2 + \eta'^2 + \zeta'^2 = z'z_0' + \zeta'^2,$$

this equation is

$$\rho'^2 + h_0 z' + h z_0' + k = 0,$$

where $-h, -h_0 = a + ib, a - ib$ respectively. The equatorial circle of this sphere is evidently given by $\zeta' = 0$, so that its equation is

$$z'z_0' + h_0 z' + h z_0' + k = 0;$$

this circle can be obtained from the equatorial circle of the sphere through P by the substitution $z' = \dfrac{\alpha z + \beta}{\gamma z + \delta}$. Hence the latter circle, by § 258, is given by

$$zz_0(\alpha\alpha_0 + h_0\alpha\gamma_0 + h\alpha_0\gamma + k\gamma\gamma_0) + z_0(\alpha_0\beta + h_0\beta\gamma_0 + h\alpha_0\delta + k\gamma_0\delta)$$
$$+ z(\alpha\beta_0 + h_0\alpha\delta_0 + h\beta_0\gamma + k\gamma\delta_0) + \beta\beta_0 + h_0\beta\delta_0 + h\beta_0\delta + k\delta\delta_0 = 0:$$

and therefore the equation of the sphere through P is

$$\rho^2(\alpha\alpha_0 + h_0\alpha\gamma_0 + h\alpha_0\gamma + k\gamma\gamma_0) + z_0(\alpha_0\beta + h_0\beta\gamma_0 + h\alpha_0\delta + k\gamma_0\delta)$$
$$+ z(\alpha\beta_0 + h_0\alpha\delta_0 + h\beta_0\gamma + k\gamma\delta_0) + \beta\beta_0 + h_0\beta\delta_0 + h\beta_0\delta + k\delta\delta_0 = 0.$$

The quantities h, h_0, k are arbitrary quantities, subject to only the single condition that the sphere passes through the point Q: there is no other relation that connects them. Hence the equation of the sphere through P must, as a condition attaching to the quantities h, h_0, k, be substantially the equivalent of the former condition given by the equation of the sphere through Q. In order that these two equations may be the same for h, h_0, k, the variables ρ'^2, z', z_0' of the point Q and those of P, being ρ^2, z, z_0, must give practically the same coefficients of h, h_0, k in the two equations, and therefore

$$\begin{aligned} \rho'^2 &: \rho^2\alpha\alpha_0 + z_0\alpha_0\beta + z\alpha\beta_0 + \beta\beta_0 \\ = z' &: \rho^2\alpha\gamma_0 + z_0\beta\gamma_0 + z\alpha\delta_0 + \beta\delta_0 \\ = z_0' &: \rho^2\alpha_0\gamma + z_0\alpha_0\delta + z\beta_0\gamma + \beta_0\delta \\ = 1 &: \rho^2\gamma\gamma_0 + z_0\gamma_0\delta + z\gamma\delta_0 + \delta\delta_0. \end{aligned}$$

These are evidently the equations which express the variables of a point Q in space in terms of the variables of the point P, when it is derived from P by the generalisation of the linear substitution

$$w' = \frac{\alpha w + \beta}{\gamma w + \delta}:$$

they may be called the equations of the substitution. It is easy to deduce that

$$\frac{\zeta'}{\zeta} = \frac{1}{\rho^2\gamma\gamma_0 + z_0\gamma_0\delta + z\gamma\delta_0 + \delta\delta_0},$$

which may be combined with the preceding equations of the substitution.

Also, the magnification for a single inversion is ds_1/ds, or r_1/r, where r_1 and r are the distances of the arcs from the centre of the sphere relative to which the inversion is effected. But $r_1/r = \zeta_1/\zeta$, where ζ_1 and ζ are the heights of the arcs above the equatorial plane; hence the magnification is ζ_1/ζ, for a single inversion. For the next inversion it is ζ_2/ζ_1, and therefore it is ζ_2/ζ for the two together; and so on. Hence the final magnification m for the whole transformation is

$$\begin{aligned} m = \frac{\zeta'}{\zeta} &= \frac{1}{\zeta^2\gamma\gamma_0 + (\gamma z + \delta)(\gamma_0 z_0 + \delta_0)} \\ &= \frac{1}{\zeta^2|\gamma|^2 + |\gamma z + \delta|^2}, \end{aligned}$$

a quantity that diminishes as the region recedes from the equatorial plane.

It is justifiable to regard the equations obtained as merely the generalisation of the substitution: they actually include the substitution in its original application to plane variables. When the variables are restricted to the plane of ξ, η, we have $\rho^2 = zz_0$, and therefore

$$z' = \frac{zz_0\alpha\gamma_0 + z_0\beta\gamma_0 + z\alpha\delta_0 + \beta\delta_0}{zz_0\gamma\gamma_0 + z_0\gamma_0\delta + z\gamma\delta_0 + \delta\delta_0} = \frac{\alpha z + \beta}{\gamma z + \delta},$$

on the removal of the factor $\gamma_0 z_0 + \delta_0$ common to the numerator and the denominator; and ζ' vanishes when $\zeta = 0$. The uniqueness of the result is an *a posteriori* justification of the initial assumption that one and the same point Q is derived from P, whatever be the inversions that are equivalent to the linear substitution.

Ex. 1. Let an elliptic substitution have u and v as its fixed points.

Draw two circles in the plane, passing through u and v and intersecting at an angle equal to half the argument of the multiplier. The transformation of the plane, caused by the substitution, is equivalent to inversions at these circles; the corresponding transformation of the space above the plane is equivalent to inversions at the spheres, having these circles as equatorial circles. It therefore follows that every point on the line of intersection of the spheres remains unchanged: hence *when a Kleinian substitution is elliptic, every point on the circle, in a plane perpendicular to the plane of x, y and having the line joining the common points of the substitution as its diameter, is unchanged by the substitution.* Poincaré calls this circle C the *double* (or *fixed*) *circle* of the elliptic substitution.

Ex. 2. Prove that, when a Kleinian substitution is hyperbolic, the only points in space, which are unchanged by it, are its double points in the plane of x, y; and shew that it changes any circle through those points into itself and also any sphere through those points into itself.

Ex. 3. Prove that, when the substitution is loxodromic, the circle C, in a plane perpendicular to the plane x, y and having as its diameter the line joining the common points of the substitution, is transformed into itself, but that the only points on the circumference left unchanged are the common points.

Ex. 4. Obtain the corresponding properties of the substitution when it is parabolic.

(All these results are due to Poincaré.)

295. The process of obtaining the division of the z-plane by means of Kleinian groups is similar to that adopted for Fuchsian groups, except that now there is no axis of real quantities or no fundamental circle conserved in that plane during the substitutions: and thus the whole plane is distributed. The polygons will be bounded by arcs of circles as before: but a polygon will not necessarily be simply connected. Multiple connectivity has already arisen in connection with real groups of the third family by taking the plane on both sides of the axis.

As there are no edges of the second kind for polygons determined by Kleinian groups, the only cycles of corners of polygons are closed cycles; let A_0, A_1, ..., A_{n-1} in order be such a cycle in a polygon R_0. Round A_0 describe a small curve, and let the successive polygons along this curve be R_0, R_1, ..., R_{n-1}, R_n,.... The corner A_0 belongs to each of these polygons: when considered as belonging to R_m, it will in that polygon be the homologue of A_m as belonging to R_0, if $m < n$; but, as belonging to R_n, it will, in that polygon, be the homologue of A_0 as belonging to R_0. Hence the substitution, which changes R_0 into R_n, has A_0 for a fixed point.

This substitution may be either elliptic or parabolic, (but not hyperbolic, § 292): that it cannot be loxodromic may be seen as follows. Let $\rho e^{i\omega}$ be the multiplier, where (§ 259) ρ is not unity and ω is not zero: and let Σ_0 denote the aggregate of polygons R_0, R_1, ..., R_{n-1}, Σ_1 the aggregate R_n, ..., R_{2n-1}, and so on. Then Σ_0 is changed to Σ_1, Σ_1 to Σ_2, and so on, by the substitution. Let p be an integer such that $p\omega \geqslant 2\pi$; then, when the substitution has been applied p times, the aggregate of the polygons is Σ_p, and it will cover the whole or part of one of the aggregates $\Sigma_0, \Sigma_1, \dots$ But, because ρ^p is not unity, Σ_p does not coincide with that aggregate or the part of that aggregate: the substitution is not then properly discontinuous, contrary to the definition of the group. Hence there is no loxodromic substitution in the group. If the substitution be elliptic, the sum of the angles of the cycle must be a submultiple of 2π; when it is parabolic, each angle of the cycle is zero.

In the generalised equations whereby points of space are transformed into one another, the plane of x, y is conserved throughout: it is natural therefore to consider the division of space on the positive side of this plane into regions $P_0, P_1, \dots$, such that P_0 is changed into all the other regions in turn by the application to it of the generalised equations. The following results can be obtained by considerations similar to those before adduced in the division of a plane*.

The boundaries of regions are either portions of spheres, having their centres in the plane of x, y, or they are portions of that plane: the regions are called polyhedral, and such boundaries are called *faces*. If the face is spherical, it is said to be of the *first kind:* if it is a portion of the plane of x, y, it is said to be of the *second kind*. Faces of the second kind, being in the plane of x, y and transformed into one another, are polygons bounded by arcs of circles.

The intersections of faces are *edges*. Again, an edge is of the *first kind*, when it is the intersection of two faces of the first kind: it is of the *second kind*, when it is the intersection of a face of the first kind with one of the second kind. An edge of the second kind is a circular arc in the plane of x, y: an edge of the first kind, being the intersection of two spheres with their centres in the plane of x, y, is a circular arc, which lies in a plane perpendicular to the plane of x, y and has its centre in that plane.

The extremities of the edges are *corners* of the polyhedra. They are of three categories:

(i) those which are above the plane of x, y and are the common extremities of at least three edges of the first kind:

(ii) those which lie in the plane of x, y and are the common extremities of at least three edges of the first kind:

(iii) those which lie in the plane of x, y and are the common extremities of at least one edge of the first kind and of at least two edges of the second kind.

* See, in particular, Poincaré, *Acta Math.*, t. iii, pp. 66 et seq.

Moreover, points at which two faces touch can be regarded as *isolated* corners, the edges of which they are the intersections not being in evidence.

Faces of a polyhedron, which are of the first kind, are conjugate in pairs: two conjugate faces are congruent by a fundamental substitution of the group.

Edges of the first kind, being the limits of the faces, arrange themselves in cycles, in the same way as the angles of a polygon in the division of the plane. If $E_0, E_1, \ldots, E_{n-1}$ be the n edges in a cycle, the number of regions which have an edge in E_0 is a multiple of n: and the sum of the dihedral angles at the edges in a cycle (the dihedral angle at an edge being the constant angle between the faces, which intersect along the edge) is a submultiple of 2π.

The relation between the polyhedral divisions of space and the polygonal divisions of the plane is as follows. Let the group be such as to cause the fundamental polyhedron P_0 to possess n faces of the second kind, say F_{01}, $F_{02}, \ldots, F_{0n}$. Every congruent polyhedron will then have n faces of the second kind; let those of P_s be $F_{s1}, F_{s2}, \ldots, F_{sn}$. Every point in the plane of x, y belongs to some one of the complete set of faces of the second kind: and, except for certain singular points and certain singular lines, no point belongs to more than one face, for the proper discontinuity of the group requires that no point of space belongs to more than one polyhedron.

Then the plane of x, y is divided into n regions, say $D_1, D_2, \ldots, D_n$; each of these regions is composed of an infinite number of polygons, consisting of the polygonal faces F. Thus D_r is composed of $F_{0r}, F_{1r}, F_{2r}, \ldots$; and these polygonal areas are such that the substitution S_s transforms F_{0r} into F_{sr}. Hence it appears that, by a Kleinian group, the whole plane is divided into a finite number of regions; and that each region is divided into an infinite number of polygons, which are congruent to one another by the substitutions of the group.

296. The preceding groups of substitutions, that have complex coefficients, have been assumed to be properly discontinuous.

Ex. Prove that, if any group of substitutions with complex coefficients be improperly discontinuous, it is improperly discontinuous only for points in the plane of x, y.

(Poincaré.)

One of the simplest and most important of the improperly discontinuous groups of substitutions, is that compounded from the three fundamental substitutions

$$z' = Sz = z + 1, \quad z' = Tz = -\frac{1}{z}, \quad z' = Vz = z + i,$$

where i has the ordinary meaning. All the substitutions are easily proved to be of the form

$$\frac{\alpha z + \beta}{\gamma z + \delta},$$