I. *Each of the two distinct pieces, into which a simply connected surface S is resolved by a cross-cut, is itself simply connected.*

If either of the pieces, made by a cross-cut ab, be not simply connected, then some cross-cut cd must be possible which will not resolve that piece into distinct portions.

If neither c nor d lie on ab, then the obliteration of the cut ab will restore the original surface S, which now is not resolved by the cut cd into distinct pieces.

If one of the extremities of cd, say c, lie on ab, then the obliteration of the portion cb will change the two pieces into a single piece which is the original surface S; and S now has a cross-cut acd, which does not resolve it into distinct pieces.

If both the extremities lie on ab, then the obliteration of that part of ab which lies between c and d will change the two pieces into one; this is the original surface S, now with a cross-cut $acdb$, which does not resolve it into distinct pieces.

These are all the possible cases should either of the distinct pieces of S not be simply connected; each of them leads to a contradiction of the simple connection of S; therefore the hypothesis on which each is based is untenable, that is, the distinct pieces of S in all the cases are simply connected.

COROLLARY 1. *A simply connected surface is resolved by n cross-cuts into $n+1$ distinct pieces, each simply connected; and an aggregate of m simply connected surfaces is resolved by n cross-cuts into $n+m$ distinct pieces each simply connected.*

COROLLARY 2. *A surface that is resolved into two distinct simply connected pieces by a cross-cut is simply connected before the resolution.*

COROLLARY 3. *If a multiply connected surface be resolved into two different pieces by a cross-cut, both of these pieces cannot be simply connected.*

We now come to a theorem* of great importance:—

II. *If a resolution of a surface by m cross-cuts into n distinct simply connected pieces be possible, and also a different resolution of the same surface by μ cross-cuts into ν distinct simply connected pieces, then $m-n=\mu-\nu$.*

Let the aggregate of the n pieces be denoted by S and the aggregate of the ν pieces by Σ: and consider the effect on the original surface of a united system of $m+\mu$ simultaneous cross-cuts made up of the two systems of the m and of the μ cross-cuts respectively. The operation of this system can be carried out in two ways: (i) by effecting the system of μ cross-cuts on S and

* The following proof of this proposition is substantially due to Neumann, p. 157. Another proof is given by Riemann, pp. 10, 11, and is amplified by Durège, *Elemente der Theorie der Functionen*, pp. 183—190; and another by Lippich, see Durège, pp. 190—197.

(ii) by effecting the system of m cross-cuts on Σ: with the same result on the original surface.

After the explanation of § 159, we may justifiably assume that the lines of the two systems of cross-cuts meet only in points, if at all: let δ be the number of points of intersection of these lines. Whenever the direction of a cross-cut meets a boundary line, the cross-cut terminates; and if the direction continue beyond that boundary line, that produced part must be regarded as a new cross-cut.

Hence the new system of μ cross-cuts applied to S is effectively equivalent to $\mu + \delta$ new cross-cuts. Before these cuts were made, S was composed of n simply connected pieces; hence, after they are applied, the new arrangement of the original surface is made up of $n + (\mu + \delta)$ simply connected pieces.

Similarly, the new system of m cross-cuts applied to Σ will give an arrangement of the original surface made up of $\nu + (m + \delta)$ simply connected pieces. These two arrangements are the same: and therefore

$$n + \mu + \delta = \nu + m + \delta,$$

so that

$$m - n = \mu - \nu.$$

It thus appears that, if by any system of q cross-cuts a multiply connected surface be resolved into a number p of pieces distinct from one another and all simply connected, the integer $q - p$ is independent of the particular system of the cross-cuts and of their configuration. The integer $q - p$ is therefore essentially associated with the character of the multiple connection of the surface: and its invariance for a given surface enables us to arrange surfaces according to the value of the integer.

No classification among the multiply connected surfaces has yet been made: they have merely been defined as surfaces in which cross-cuts can be made that do not resolve the surface into distinct pieces.

It is natural to arrange them in classes according to the number of cross-cuts which are necessary to resolve the surface into one of simple connection or a number of pieces each of simple connection.

For a simply connected surface, no such cross-cut is necessary: then $q = 0$, $p = 1$, and in general $q - p = -1$. We shall say that the *connectivity** is unity. Examples are furnished by the area of a plane circle, and by a spherical surface with one hole†.

A surface is called doubly-connected when, by one appropriate cross-cut, the surface is changed into a single surface of simple connection: then $q = 1$, $p = 1$ for this particular resolution, and therefore in general, $q - p = 0$. We

* Sometimes *order of connection*, sometimes *adelphic order;* the German word, that is used, is *Grundzahl*.

† The hole is made to give the surface a boundary (§ 163).

shall say that the connectivity is 2. Examples are furnished by a plane ring and by a spherical surface with two holes.

A surface is called triply-connected when, by two appropriate cross-cuts, the surface is changed into a single surface of simple connection: then $q = 2$, $p = 1$ for this particular resolution and therefore, in general, $q - p = 1$. We shall say that the connectivity is 3. Examples are furnished by the surface of an anchor-ring with one hole in it*, and by the surfaces† in Figure 39, the surface in (2) not being in one plane but one part beneath another.

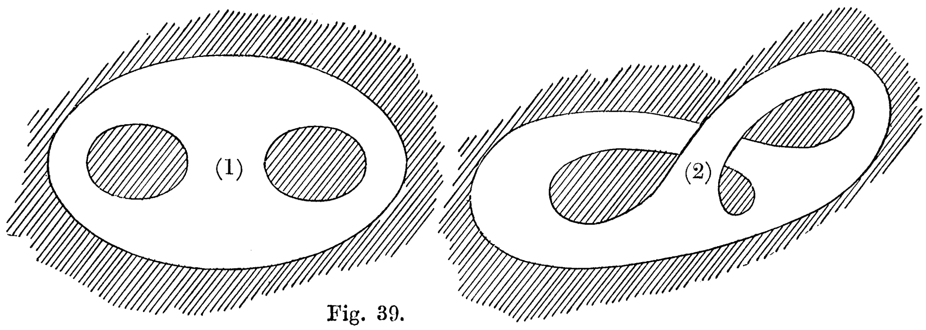

Fig. 39.

And, in general, a surface will be said to be N-ply connected or its connectivity will be denoted by N, if, by $N - 1$ appropriate cross-cuts, it can be changed into a single surface that is simply connected‡. For this particular resolution $q = N - 1$, $p = 1$: and therefore in general

$$q - p = N - 2,$$

or

$$N = q - p + 2.$$

Let a cross-cut l be drawn in a surface of connectivity N. There are two cases to be considered, according as it does not or does divide the surface into distinct pieces.

First, let the surface be only one piece after l is drawn: and let its connectivity then be N'. If in the original surface q cross-cuts (one of which can, after the preceding proposition, be taken to be l) be drawn dividing the surface into p simply connected pieces, then

$$N = q - p + 2.$$

To obtain these p simply connected pieces from the surface after the cross-cut l, it is evidently sufficient to make the $q - 1$ original cross-cuts other than l; that is, the modified surface is such that by $q - 1$ cross-cuts it is resolved into p simply connected pieces, and therefore

$$N' = (q - 1) - p + 2.$$

Hence $N' = N - 1$, or the connectivity of the surface is diminished by unity.

* The hole is made to give the surface a boundary (§ 163).

† Riemann, p. 89.

‡ A few writers estimate the connectivity of such a surface as $N - 1$, the same as the number of cross-cuts which can change it into a single surface of the simplest rank of connectivity: the estimate in the text seems preferable.

Secondly, let the surface be two pieces after l is drawn, of connectivities N_1 and N_2 respectively. Let the appropriate $N_1 - 1$ cross-cuts in the former, and the appropriate $N_2 - 1$ in the latter, be drawn so as to make each a simply connected piece. Then, together, there are two simply connected pieces.

To obtain these two pieces from the original surface, it will suffice to make in it the cross-cut l, the $N_1 - 1$ cross-cuts, and the $N_2 - 1$ cross-cuts, that is, $1 + (N_1 - 1) + (N_2 - 1)$ or $N_1 + N_2 - 1$ cross-cuts in all. Since these, when made in the surface of connectivity N, give two pieces, we have

$$N = (N_1 + N_2 - 1) - 2 + 2,$$

and therefore

$$N_1 + N_2 = N + 1.$$

If one of the pieces be simply connected, the connectivity of the other is N; so that, if a simply connected piece of surface be cut off a multiply connected surface, the connectivity of the remainder is unchanged. Hence:

III. *If a cross-cut be made in a surface of connectivity N and if it do not divide it into separate pieces, the connectivity of the modified surface is $N - 1$; but if it divide the surface into two separate pieces of connectivities N_1 and N_2, then $N_1 + N_2 = N + 1$.*

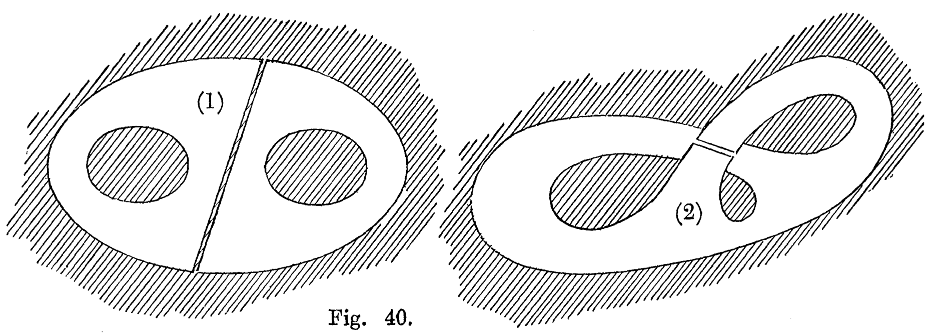

Fig. 40.

Illustrations are shewn, in Fig. 40, of the effect of cross-cuts on the two surfaces in Fig. 39.

IV. In the same way it may be proved that, *if s cross-cuts be made in a surface of connectivity N and divide it into $r + 1$ separate pieces (where $r \leqslant s$) of connectivities $N_1, N_2, \ldots, N_{r+1}$ respectively, then*

$$N_1 + N_2 + \ldots + N_{r+1} = N + 2r - s,$$

a more general result including both of the foregoing cases.

Thus far we have been considering only cross-cuts: it is now necessary to consider loop-cuts, so far as they affect the connectivity of a surface in which they are made.

A loop-cut is changed into a cross-cut, if from A any point of it a cross-cut be made to any point C in a boundary-curve of the original surface, for $CAbdA$ (Fig. 41) is then evidently a cross-cut of the original surface; and CA is a cross-cut of the surface, which is the modification of the original surface after the loop-cut has been made. Since, by definition, a loop-cut does not meet the boundary, the cross-cut CA does not divide the modified surface into distinct pieces; hence, according as the effect of the loop-cut is, or is not, that of making distinct pieces, so will the effect of the whole cross-cut be, or not be, that of making distinct pieces.

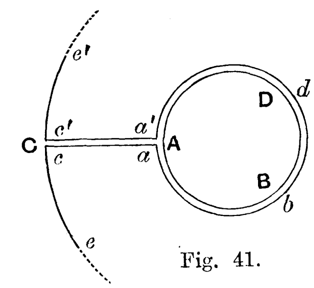

Fig. 41.

161. Let a loop-cut be drawn in a surface of connectivity N; as before for a cross-cut, there are two cases for consideration, according as the loop-cut does or does not divide the surface into distinct pieces.

First, let it divide the surface into two distinct pieces, say of connectivities N_1 and N_2 respectively. Change the loop-cut into a cross-cut of the original surface by drawing a cross-cut in either of the pieces, say the second, from a point in the course of the loop-cut to some point of the original boundary. This cross-cut, as a section of that piece, does not divide it into distinct pieces: and therefore the connectivity is now $N_2' \,(= N_2 - 1)$. The effect of the whole section, which is a single cross-cut, of the original surface is to divide it into two pieces, the connectivities of which are N_1 and N_2': hence, by § 160, III.,

$$N_1 + N_2' = N + 1,$$

and therefore $$N_1 + N_2 = N + 2.$$

If the piece cut out be simply connected, say $N_1 = 1$, then the connectivity of the remainder is $N + 1$. But such a removal of a simply connected piece by a loop-cut is the same as making a hole in a continuous part of the surface: and therefore *the effect of making a simple hole in a continuous part of a surface is to increase by unity the connectivity of the surface.*

If the piece cut out be doubly connected, say $N_1 = 2$, then the connectivity of the remainder is N, the same as the connectivity of the original surface. Such a portion would be obtained by cutting out a piece with a hole in it which, so far as concerns the original surface, would be the same as merely enlarging the hole—an operation that naturally would not affect the connectivity.

Secondly, let the loop-cut not divide the surface into two distinct pieces: and let N' be the connectivity of the modified surface. In this modified surface make a cross-cut k from any point of the loop-cut to a point of the boundary: this does not divide it into distinct pieces and therefore the connectivity after this last modification is $N' - 1$. But the surface thus finally modified is derived from the original surface by the single cross-cut, constituted by the combination of k with the loop-cut: this single cross-cut does not divide the surface into distinct pieces and therefore the connectivity after the modification is $N-1$. Hence

$$N'-1=N-1,$$

that is, $N'=N$, or *the connectivity of a surface is not affected by a loop-cut which does not divide the surface into distinct pieces.*

Both of these results are included in the following theorem:—

V. *If after any number of loop-cuts made in a surface of connectivity N, there be $r+1$ distinct pieces of surface, of connectivities $N_1, N_2, \ldots, N_{r+1}$, then*

$$N_1+N_2+\ldots\ldots+N_{r+1}=N+2r.$$

Let the number of loop-cuts be s. Each of them can be changed into a cross-cut of the original surface, by drawing in some one of the pieces, as may be convenient, a cross-cut from a point of the loop-cut to a point of a boundary; this new cross-cut does not divide the piece in which it is drawn into distinct pieces. If k such cross-cuts (where k may be zero) be drawn in the piece of connectivity N_m, the connectivity becomes N_m', where

$$N_m'=N_m-k;$$

hence $$\sum_{m=1}^{r+1} N_m'=\sum_{m=1}^{r+1} N_m-\Sigma k=\sum_{m=1}^{r+1} N_m-s.$$

We now have s cross-cuts dividing the surface of connectivity N into $r+1$ distinct pieces, of connectivities $N_1', N_2', \ldots, N_r', N_{r+1}'$; and therefore, by § 160, IV.,

$$N_1'+\ldots+N_r'+N_{r+1}'=N+2r-s,$$

so that $$N_1+N_2+\ldots+N_{r+1}=N+2r.$$

This result could have been obtained also by combination and repetition of the two results obtained for a single loop-cut.

Thus a spherical surface with one hole in it is simply connected: when $n-1$ other different holes* are made in it, the edges of the holes being outside one another, the connectivity of the surface is increased by $n-1$, that is, it becomes n. Hence *a spherical surface with n holes in it is n-ply connected.*

162. Occasionally, it is necessary to consider the effect of a slit made in the surface.

If the slit have neither of its extremities on a boundary (and therefore no point on a boundary) it can be regarded as the limiting form of a loop-cut which makes a hole in the surface. Such a slit therefore (§ 161) increases the connectivity by unity.

* These are holes in the surface, not holes bored through the volume of the sphere; one of the latter would give two holes in the surface.

If the slit have one extremity (but no other point) on a boundary, it can be regarded as the limiting form of a cross-cut, which returns on itself as in the figure, and cuts off a single simply connected piece. Such a slit therefore (§ 160, III.) leaves the connectivity unaltered.

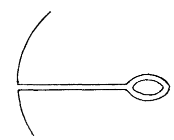

Fig. 42.

If the slit have both extremities on boundaries, it ceases to be merely a slit: it is a cross-cut the effect of which on the connectivity has been obtained. We do not regard such sections as slits.

163. In the preceding investigations relative to cross-cuts and loop-cuts, reference has continually been made to the boundary of the surface considered.

The *boundary* of a surface consists of a line returning to itself, or of a system of lines each returning to itself. Each part of such a boundary-line as it is drawn is considered a part of the boundary, and thus a boundary-line cannot cut itself and pass beyond its earlier position, for a boundary cannot be crossed: each boundary-line must therefore be a simple curve*.

Most surfaces have boundaries: an exception arises in the case of closed surfaces whatever be their connectivity. It was stated (§ 159) that a boundary is assigned to such a surface by drawing an infinitesimal simple curve in it or, what is the same thing, by making a small hole. The advantage of this can be seen from the simple example of a spherical surface.

When a small hole is made in any surface the connectivity is increased by unity: the connectivity of the spherical surface after the hole is made is unity, and therefore the connectivity of the complete spherical surface must be taken to be zero.

The mere fact that the connectivity is less than unity, being that of the simplest connected surfaces with which we have to deal, is not in itself of importance. But let us return for a moment to the suggested method of determining the connectivity by means of the evanescence of circuits without crossing the boundary. When the surface is the complete spherical surface (Fig. 43), there are two essentially distinct ways of making a circuit C evanescent, first, by making it collapse into the point a, secondly by making it expand over the equator and then collapse into the point b. One of the two is superfluous: it introduces an element of doubt as to the mode of evanescence unless that mode be specified—a specification which in itself is tantamount to an assignment of

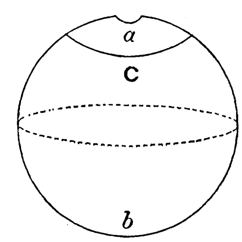

Fig. 43.

* Also a line not returning to itself may be a boundary; it can be regarded as the limit of a simple curve when the area becomes infinitesimal.

boundary. And in the case of multiply connected surfaces the absence of boundary, as above, leads to an artificial reduction of the connectivity by unity, arising not from the greater simplicity of the surface but from the possibility of carrying out in two ways the operation of reducing any circuit to given circuits, which is most effective when only one way is permissible. We shall therefore assume a boundary assigned to such closed surfaces as in the first instance are destitute of boundary.

164. The relations between the number of boundaries and the connectivity of a surface are given by the following propositions.

I. *The boundary of a simply connected surface consists of a single line.*

When a boundary consists of separate lines, then a cross-cut can be made from a point of one to a point of another. By proceeding from P, a point on one side of the cross-cut, along the boundary $ac \ldots c'a'$ we can by a line lying wholly in the surface reach a point Q on the other side of the cross-cut: hence the parts of the surface on opposite sides of the cross-cut are connected. The surface is therefore not resolved into distinct pieces by the cross-cut.

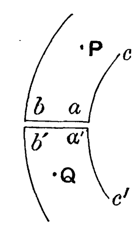

Fig. 44.

A simply connected surface is resolved into distinct pieces by each cross-cut made in it: such a cross-cut as the foregoing is therefore not possible, that is, there are not separate lines which make up its boundary. It has a boundary: the boundary therefore consists of a single line.

II. *A cross-cut either increases by unity or diminishes by unity the number of distinct boundary-lines of a multiply connected surface.*

A cross-cut is made in one of three ways: either from a point a of one boundary-line A to a point b of another boundary-line B; or from a point a of a boundary-line to another point a' of the same boundary-line; or from a point of a boundary-line to a point in the cut itself.

If made in the first way, a combination of one edge of the cut, the remainder of the original boundary A, the other edge of the cut and the remainder of the original boundary B taken in succession, form a single piece of boundary; this replaces the two boundary-lines A and B which existed distinct from one another before the cross-cut was made. Hence the number of lines is diminished by unity. An example is furnished by a plane ring (ii., Fig. 37, p. 314).

If made in the second way, the combination of one edge of the cut with the piece of the boundary on one side of it makes one boundary-line, and the combination of the other edge of the cut with the other piece of the boundary makes another boundary-line. Two boundary-lines, after the cut is made, replace a single boundary-line, which existed before it was made: hence the number of lines is increased by unity. Examples are furnished by the cut surfaces in Fig. 40, p. 319.

If made in the third way, the cross-cut may be considered as constituted by a loop-cut and a cut joining the loop-cut to the boundary. The boundary-lines may now be considered as constituted (Fig. 41, p. 320) by the closed curve ABD and the closed boundary $abda'c'e'\ldots eca$; that is, there are now two boundary-lines instead of the single boundary-line $ce\ldots e'c'c$ in the uncut surface. Hence the number of distinct boundary-lines is increased by unity.

COROLLARY. *A loop-cut increases the number of distinct boundary-lines by two.*

This result follows at once from the last discussion.

III. *The number of distinct boundary-lines of a surface of connectivity N is $N-2k$, where k is a positive integer that may be zero.*

Let m be the number of distinct boundary-lines; and let $N-1$ appropriate cross-cuts be drawn, changing the surface into a simply connected surface. Each of these cross-cuts increases by unity or diminishes by unity the number of boundary-lines; let these units of increase or of decrease be denoted by $\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}$. Each of the quantities ϵ is ± 1; let k of them be positive, and $N-1-k$ negative. The total number of boundary-lines is therefore

$$m + k - (N - 1 - k).$$

The surface now is a single simply connected surface, and there is therefore only one boundary-line; hence

$$m + k - (N - 1 - k) = 1,$$

so that

$$m = N - 2k;$$

and evidently k is an integer that may be zero.

COROLLARY 1. *A closed surface with a single boundary-line* is of odd connectivity.*

For example, the surface of an anchor-ring, when bounded, is of connectivity 3; the surface, obtained by boring two holes through the volume of a solid sphere, is, when bounded, of connectivity 5.

If the connectivity of a closed surface with a single boundary be $2p+1$, the surface is often said† to be of *class* p (§ 178, p. 349.)

COROLLARY 2. *If the number of distinct boundary lines of a surface of connectivity N be N, any loop-cut divides the surface into two distinct pieces.*

After the loop-cut is made, the number of distinct boundary-lines is $N+2$; the connectivity of the whole of the cut surface is therefore not less

* See § 159.

† The German word is *Geschlecht;* French writers use the word *genre*, and Italians *genere*.

than $N+2$. It has been proved that a loop-cut, which does not divide the surface into distinct pieces, does not affect the connectivity; hence as the connectivity has been increased, the loop-cut must divide the surface into two distinct pieces. It is easy, by the result of § 161, to see that, after the loop-cut is made, the sum of connectivities of the two pieces is $N+2$, so that the connectivity of the whole of the cut surface is equal to $N+2$.

Note. Throughout these propositions, a tacit assumption has been made, which is important for this particular proposition when the surface is the means of representing the variable. The assumption is that *the surface is bifacial and not unifacial*; it has existed implicitly throughout all the geometrical representations of variability: it found explicit expression in § 4 when the plane was brought into relation with the sphere: and a cut in a surface has been counted a single cut, occurring in one face, though it would have to be counted as two cuts, one on each side, were the surface unifacial.

The propositions are not necessarily valid, when applied to unifacial surfaces. Consider a surface made out of a long rectangular slip of paper, which is twisted once (or any odd number of times) and then has its ends fastened together. This surface is of double connectivity, because one section can be made across it which does not divide it into separate pieces; it has only a *single* boundary-line, so that Prop. III. just proved does not apply. The surface is unifacial; and it is possible, without meeting the boundary, to pass continuously in the surface from a point P to another point Q which could be reached merely by passing through the material at P.

We therefore do not retain unifacial surfaces for consideration.

165. The following proposition, substantially due to Lhuilier*, may be taken in illustration of the general theory.

If a closed surface of connectivity $2N+1$ (or of class N) be divided by circuits into any number of simply connected portions, each in the form of a curvilinear polygon, and if F be the number of polygons, E be the number of edges and S the number of angular points, then

$$2N = 2 + E - F - S.$$

Let the edges E be arranged in systems, a system being such that any line in it can be reached by passage along some other line or lines of the system; let k be the number of such systems†. To resolve the surface into a number of simply connected pieces composed of the F polygons, the cross-cuts will be made along the edges; and therefore, unless a boundary be assigned

* Gergonne, *Ann. de Math.*, t. iii, (1813), pp. 181—186; see also Möbius, *Ges. Werke*, t. ii, p. 468. A *circuit* is defined in § 166.

† The value of k is 1 for the proposition and is greater than 1 for the Corollary.

to the surface in each system of lines, the first cut for any system will be a loop-cut. We therefore take k points, one in each system as a boundary; the first will be taken as the natural boundary of the surface, and the remaining $k-1$, being the limiting forms of $k-1$ infinitesimal loop-cuts, increase the connectivity of the surface by $k-1$, that is, the connectivity now is $2N+k$.

The result of the cross-cuts is to leave F simply connected pieces: hence Q, the number of cross-cuts, is given by

$$Q = 2N + k + F - 2.$$

At every angular point on the uncut surface, three or more polygons are contiguous. Let S_m be the number of angular points, where m polygons are contiguous; then

$$S = S_3 + S_4 + S_5 + \ldots$$

Again, the number of edges meeting at each of the S_3 points is three, at each of the S_4 points is four, at each of the S_5 points is five, and so on; hence, in taking the sum $3S_3 + 4S_4 + 5S_5 + \ldots$, each edge has been counted twice, once for each extremity. Therefore

$$2E = 3S_3 + 4S_4 + 5S_5 + \ldots$$

Consider the composition of the extremities of the cross-cuts; the number of the extremities is $2Q$, twice the number of cross-cuts.

Each of the k points furnishes two extremities; for each such point is a boundary on which the initial cross-cut for each of the systems must begin and must end. These points therefore furnish $2k$ extremities.

The remaining extremities occur in connection with the angular points. In making a cut, the direction passes from a boundary along an edge, past the point along another edge and so on, until a boundary is reached; so that on the first occasion when a cross-cut passes through a point, it is made along two of the edges meeting at the point. Every other cross-cut passing through that point must begin or end there, so that each of the S_3 points will furnish one extremity (corresponding to the remaining one cross-cut through the point), each of the S_4 points will furnish two extremities (corresponding to the remaining two cross-cuts through the point), and so on. The total number of extremities thus provided is

$$S_3 + 2S_4 + 3S_5 + \ldots$$

Hence
$$2Q = 2k + S_3 + 2S_4 + 3S_5 + \ldots$$
$$= 2k + 2E - 2S,$$

or
$$Q = k + E - S,$$

which combined with
$$Q = 2N + k + F - 2,$$

leads to the relation
$$2N = 2 + E - F - S.$$

The simplest case is that of a sphere, when Euler's relation $F+S=E+2$ is obtained. The case next in simplicity is that of an anchor-ring, for which the relation is $F+S=E$.

COROLLARY. *If the result of making the cross-cuts along the various edges be to give the F polygons, not simply connected areas but areas of connectivities N_1+1, N_2+1, ..., N_F+1 respectively, then the connectivity of the original surface is given by*

$$2N = 2 + E - F - S + \sum_{r=1}^{F} N_r.$$

166. The method of determining the connectivity of a surface by means of a system of cross-cuts, which resolve it into one or more simply connected pieces, will now be brought into relation with the other method, suggested in § 159, of determining the connectivity by means of irreducible circuits.

A closed line drawn on the surface is called a *circuit*.

A circuit, which can be reduced to a point by continuous deformation without crossing the boundary, is called *reducible*; a circuit, which cannot be so reduced, is called *irreducible*.

An irreducible circuit is either (i) *simple*, when it cannot without crossing the boundary be deformed continuously into repetitions of one or more circuits; or (ii) *multiple*, when it can without crossing the boundary be deformed continuously into repetitions of a single circuit; or (iii) *compound*, when it can without crossing the boundary be deformed continuously into combinations of different circuits, that may be simple or multiple. The distinction between simple circuits and compound circuits, that involve no multiple circuits in their combination, depends upon conventions adopted for each particular case.

A circuit is said to be *reconcileable* with the system of circuits into a combination of which it can be continuously deformed.

If a system of circuits be reconcileable with a reducible circuit, the system is said to be reducible.

As there are two directions, one positive and the other negative, in which a circuit can be described, and as there are possibilities of repetitions and of compositions of circuits, it is clear that circuits can be represented by linear algebraical expressions involving real quantities and having merely numerical coefficients.

Thus a reducible circuit can be denoted by 0.

If a simple irreducible circuit, positively described, be denoted by a, the same circuit, negatively described, can be denoted by $-a$.

The multiple circuit, which is composed of m positive repetitions of the simple irreducible circuit a, would be denoted by ma; but if the m repetitions were negative, the multiple circuit would be denoted by $-ma$.

A compound circuit, reconcileable with a system of simple irreducible circuits $a_1, a_2, \ldots, a_n$ would be denoted by $m_1a_1 + m_2a_2 + \ldots + m_na_n$, where $m_1, m_2, \ldots, m_n$ are positive or negative integers, being the net number of positive or negative descriptions of the respective simple irreducible circuits.

The condition of the reducibility of a system of circuits $a_1, a_2, \ldots, a_n$, each one of which is simple and irreducible, is that integers $m_1, m_2, \ldots, m_n$ should exist such that

$$m_1a_1 + m_2a_2 + \ldots + m_na_n = 0,$$

the sign of equality in this equation, as in other equations, implying that continuous deformation without crossing the boundary can change into one another the circuits, denoted by the symbols on either side of the sign.

The representation of any compound circuit in terms of a system of independent irreducible circuits is unique: if there were two different expressions, they could be equated in the foregoing sense and this would imply the existence of a relation

$$p_1a_1 + p_2a_2 + \ldots + p_na_n = 0,$$

which is excluded by the fact that the system is irreducible.

Further, equations can be combined linearly, provided that the coefficients of the combinations be merely numerical.

167. In order, then, to be in a position to estimate circuits on a multiply connected surface, it is necessary that an irreducible system of irreducible simple circuits should be known, such a system being considered complete when every other circuit on the surface is reconcileable with the system.

Such a system is not necessarily unique; and it must be proved that, *if more than one complete system be obtainable, any circuit can be reconciled with each system.*

First, *the number of simple irreducible circuits in any complete system must be the same for the same surface.*

Let $a_1, \ldots, a_p$; and $b_1, \ldots, b_n$; be two complete systems. Because $a_1, \ldots, a_p$ constitute a complete system, every circuit of the system of circuits b is reconcileable with it; that is, integers m_{ij} exist, such that

$$b_r = m_{1r}a_1 + m_{2r}a_2 + \ldots + m_{pr}a_p,$$

for $r = 1, 2, \ldots, n$. If n were $> p$, then by combining linearly each equation after the first p equations with those p equations, and eliminating $a_1, \ldots, a_p$ from the set of $p + 1$ equations, we could derive $n - p$ relations of the form

$$M_1b_1 + M_2b_2 + \ldots + M_nb_n = 0,$$

where the coefficients M, being determinants the constituents of which are integers, would be integers. The system of circuits b is irreducible, and there are therefore no such relations; hence n is not greater than p.

Similarly, by considering the reconciliation of each circuit a with the irreducible system of circuits b, it follows that p is not greater than n.

Hence p and n are equal to one another. And, because each system is a complete system, there are integers A and B such that

$$\left.\begin{array}{l} a_r = A_{r1}b_1 + A_{r2}b_2 + \dots + A_{rn}b_n \quad (r = 1, \dots, n) \\ b_s = B_{s1}a_1 + B_{s2}a_2 + \dots + B_{sn}a_n \quad (s = 1, \dots, n) \end{array}\right\}.$$

The determinant of the integers A is equal to ± 1; likewise the determinant of the integers B.

Secondly, *let x be a circuit reconcileable with the system of circuits a: it is reconcileable with any other complete system of circuits.*

Since x is reconcileable with the system a, integers $m_1, \dots, m_n$ can be found such that

$$x = m_1a_1 + \dots + m_na_n.$$

Any other complete system of n circuits b is such that the circuits a can be expressed in the form

$$a_r = A_{r1}b_1 + \dots + A_{rn}b_n, \quad (r = 1, \dots, n),$$

where the coefficients A are integers; and therefore

$$\begin{aligned} x &= b_1 \sum_{r=1}^{n} m_rA_{r1} + b_2 \sum_{r=1}^{n} m_rA_{r2} + \dots + b_n \sum_{r=1}^{n} m_rA_{rn} \\ &= q_1b_1 + q_2b_2 + \dots + q_nb_n, \end{aligned}$$

where the coefficients q are integers, that is, x is reconcileable with the complete system of circuits b.

168. It thus appears that for the construction of any circuit on a surface, it is sufficient to know some one complete system of simple irreducible circuits. A complete system is supposed to contain the smallest possible number of simple circuits: any one which is reconcileable with the rest is omitted, so that the circuits of a system may be considered as independent. Such a system is indicated by the following theorems:—

I. *No irreducible simple circuit can be drawn on a simply connected surface*.*

If possible, let an irreducible circuit C be drawn in a simply connected surface with a boundary B. Make a loop-cut along C, and change it into a cross-cut by making a cross-cut A from some point of C to a point of B; this cross-cut divides the surface into two simply connected pieces, one of which is bounded by B, the two edges of A, and one edge of the cut along C, and the other of which is bounded entirely by the cut along C.

The latter surface is smaller than the original surface; it is simply connected and has a single boundary. If an irreducible simple circuit can be drawn on it, we proceed as before, and again obtain a still smaller simply connected surface. In this way, we ultimately obtain an infinitesimal

* All surfaces considered are supposed to be bounded.

element; for every cut divides the surface, in which it is made, into distinct pieces. Irreducible circuits cannot be drawn in this element; and therefore its boundary is reducible. This boundary is a circuit in a larger portion of the surface: the circuit is reducible so that, in that larger portion no irreducible circuit is possible and therefore its boundary is reducible. This boundary is a circuit in a still larger portion, and the circuit is reducible: so that in this still larger portion no irreducible circuit is possible and once more the boundary is reducible.

Proceeding in this way, we find that no irreducible simple circuit is possible in the original surface.

COROLLARY. *No irreducible circuit can be drawn on a simply connected surface.*

II. *A complete system of irreducible simple circuits for a surface of connectivity N contains $N-1$ simple circuits, so that every other circuit on the surface is reconcileable with that system.*

Let the surface be resolved by cross-cuts into a single simply connected surface: $N-1$ cross-cuts will be necessary. Let CD be any one of them: and let a and b be two points on the opposite edges of the cross-cut. Then since the surface is simply connected, a line can be drawn in the surface from a to b without passing out of the surface or without meeting a part of the boundary, that is, without meeting any other cross-cut. The cross-cut CD ends either in another cross-cut or in a boundary; the line $ae\ldots fb$ surrounds that other cross-cut or that boundary as the case may be: hence, if the cut CD be obliterated, the line $ae\ldots fba$ is irreducible on the surface in which the other $N-2$ cross-cuts are made. But it meets none of those cross-cuts; hence, when they are all obliterated so as to restore the unresolved surface of connectivity N, it is an irreducible circuit. It is evidently not a repeated circuit; hence it is an irreducible simple circuit. Hence *the line of an irreducible simple circuit on an unresolved surface is given by a line passing from a point on one edge of a cross-cut in the resolved surface to a point on the opposite edge.*

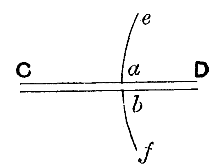

Fig. 45.

Since there are $N-1$ cross-cuts, it follows that $N-1$ irreducible simple circuits can thus be obtained: one being derived in the foregoing manner from each of the cross-cuts, which are necessary to render the surface simply connected. It is easy to see that each of the irreducible circuits on an unresolved surface is, by the cross-cuts, rendered impossible as a circuit on the resolved surface.

But every other irreducible circuit C is reconcileable with the $N-1$ circuits, thus obtained. If there be one not reconcileable with these $N-1$ circuits, then, when all the cross-cuts are made, the circuit C is not rendered impossible, if it be not reconcileable with those which are rendered impossible by the cross-cuts: that is, there is on the resolved surface an irreducible circuit. But the resolved surface is simply connected, and therefore no irreducible circuit can be drawn on it: hence the hypothesis as to C, which leads to this result, is not tenable.

Thus every other circuit is reconcileable with the system of $N-1$ circuits: and therefore *the system is complete**.

This method of derivation of the circuits at once indicates how far a system is arbitrary. Each system of cross-cuts leads to a complete system of irreducible simple circuits, and vice versa; as the one system is not unique, so the other system is not unique.

For the general question, Jordan's memoir, *Des contours tracés sur les surfaces*, Liouville, 2me Sér., t. xi., (1866), pp. 110—130, may be consulted.

Ex. 1. On a doubly connected surface, one irreducible simple circuit can be drawn. It is easily obtained by first resolving the surface into one that is simply connected—a single cross-cut CD is effective for this purpose—and then by drawing a curve aeb in the

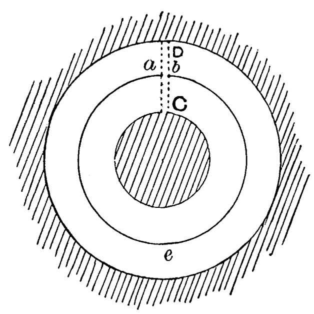

Fig. 46, (i).

surface from one edge of the cross-cut to the other. All other irreducible circuits on the unresolved surface are reconcileable with the circuit *aeba*.

Ex. 2. On a triply-connected surface, two independent irreducible circuits can be

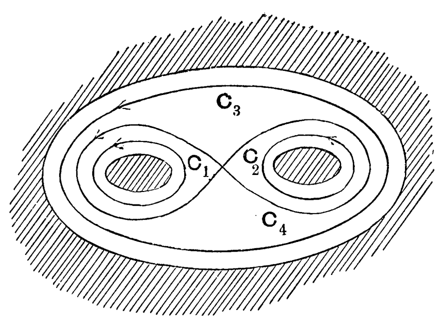

Fig. 46, (ii).

* If the number of independent irreducible simple circuits be adopted as a basis for the definition of the connectivity of a surface, the result of the proposition would be taken as the definition: and the resolution of the surface into one, which is simply connected, would then be obtained by developing the preceding theory in the reverse order.

drawn. Thus in the figure C_1 and C_2 will form a complete system. The circuits C_3 and C_4 are also irreducible : they can evidently be deformed into C_1 and C_2 and reducible circuits by continuous deformation : in the algebraical notation adopted, we have

$$C_3 = C_1 + C_2, \quad C_4 = C_1 - C_2.$$

Ex. 3. Another example of a triply connected surface is given in Fig. 47. Two irreducible simple circuits are C_1 and C_2. Another irreducible circuit is C_3; this can be

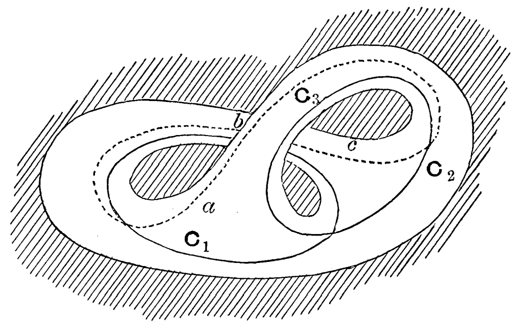

Fig. 47.

reconciled with C_1 and C_2 by drawing the point a into coincidence with the intersection of C_1 and C_2, and the point c into coincidence with the same point.

Ex. 4. As a last example, consider the surface of a solid sphere with n holes bored through it. The connectivity is $2n+1$: hence $2n$ independent irreducible simple circuits

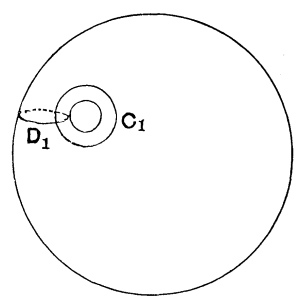

Fig. 48.

can be drawn on the surface. The simplest complete system is obtained by taking $2n$ curves: made up of a set of n, each round one hole, and another set of n, each through one hole.

A resolution of this surface is given by taking cross-cuts, one round each hole (making the circuits through the holes no longer possible) and one through each hole (making the circuits round the holes no longer possible).

The simplest case is that for which $n=1$: the surface is equivalent to the anchor-ring.

169. Surfaces are at present being considered in view of their use as a means of representing the value of a complex variable. The foregoing investigations imply that surfaces can be classed according to their connectivity; and thus, having regard to their designed use, the question arises as to whether all surfaces of the same connectivity are equivalent to one another, so as to be transformable into one another.

Moreover, a surface can be physically deformed and still remain suitable for representation of the variable, provided certain conditions are satisfied. We thus consider geometrical transformation as well as physical deformation; but we are dealing only with the general results and not with the mathematical relations of stretching and bending, which are discussed in treatises on Analytical Geometry*.

It is evident that continuity is necessary for both: discontinuity would imply discontinuity in the representation of the variable. Points that are contiguous (that is, separated only by small distances measured in the surface) must remain contiguous†: and one point in the unchanged surface must correspond to only one point in the changed surface. Hence *in the continuous deformation of a surface there may be stretching and there may be bending; but there must be no tearing and there must be no joining.*

For instance, a single untwisted ribbon, if cut, comes to be simply connected. If a twist through 180° be then given to one end and that end be then joined to the other, we shall have a once-twisted ribbon, which is a surface with only one face and only one edge; it cannot be looked upon as an equivalent of the former surface.

A spherical surface with a single hole can have the hole stretched and the surface flattened, so as to be the same as a bounded portion of a plane: the two surfaces are equivalent to one another. Again, in the spherical surface, let a large indentation be made: let both the outer and the inner surfaces be made spherical; and let the mouth of the indentation be contracted into the form of a long, narrow hole along a part of a great circle. When each point of the inner surface is geometrically moved so that it occupies the position of its reflexion in the diametral plane of the hole, the final form§ of the whole surface is that of a two-sheeted surface with a junction along a line: it is a spherical winding-surface, and is equivalent to the simply connected spherical surface.

170. It is sufficient, for the purpose of representation, that the two surfaces should have a point-to-point transformation: it is not necessary that physical deformation, without tears or joins, should be actually possible. Thus a ribbon with an even number of twists would be as effective as a limited portion of a cylinder, or (what is the same thing) an untwisted ribbon: but it is not possible to deform the one into the other physically‡.

It is easy to see that either deformation or transformation of the kind considered *will change a bifacial surface into a bifacial surface;* that *it will not alter the connectivity,* for it will not change irreducible circuits into

* See, for instance, Frost's *Solid Geometry*, (3rd ed.), pp. 342—352.

† Distances between points must be measured along the surface, not through space; the distance between two points is a length which one point would traverse before reaching the position of the other, the motion of the point being restricted to take place in the surface. Examples will arise later, in Riemann's surfaces, in which points that are contiguous in space are separated by finite distances on the surface.

§ Clifford, *Coll. Math. Papers*, p. 250.

‡ The difference between the two cases is that, in physical deformation, the surfaces are the surfaces of continuous matter and are impenetrable; while, in geometrical transformation, the surfaces may be regarded as penetrable without interference with the continuity.

reducible circuits, and the number of independent irreducible circuits determines the connectivity: and that *it will not alter the number of boundary curves*, for a boundary will be changed into a boundary. These are necessary relations between the two forms of the surface: it is not difficult to see that they are sufficient for correspondence. For if, on each of two bifacial surfaces with the same number of boundaries and of the same connectivity, a complete system of simple irreducible circuits be drawn, then, when the members of the systems are made to correspond in pairs, the full transformation can be effected by continuous deformation of those corresponding irreducible circuits. It therefore follows that:—

The necessary and sufficient conditions, that two bifacial surfaces may be equivalent to one another for the representation of a variable, are that the two surfaces should be of the same connectivity and should have the same number of boundaries.

As already indicated, this equivalence is a geometrical equivalence: deformation may be (but is not of necessity) physically possible.

Similarly, the presence of one or of several knots in a surface makes no essential difference in the use of the surface for representing a variable. Thus a long cylindrical surface is changed into an anchor-ring when its ends are joined together; but the changed surface would be equally effective for purposes of representation if a knot were tied in the cylindrical surface before the ends are joined.

But it need hardly be pointed out that though surfaces, thus twisted or knotted, are equivalent for the purpose indicated, they are not equivalent for all topological enumerations.

Seeing that bifacial surfaces, with the same connectivity and the same number of boundaries, are equivalent to one another, it is natural to adopt, as the surface of reference, some simple surface with those characteristics; thus for a surface of connectivity $2p+1$ with a single boundary, the surface of a solid sphere, bounded by a point and pierced through with p holes, could be adopted.

Klein calls* such a surface of reference a *Normal Surface.*

It has been seen that a bounded spherical surface and a bounded simply connected part of a plane are equivalent—they are, moreover, physically deformable into one another.

An untwisted closed ribbon is equivalent to a bounded piece of a plane with one hole in it—they are deformable into one another: but if the ribbon, previous to being closed, have undergone an even number of twists each through 180°, they are still equivalent but are not physically deformable into one another. Each of the bifacial surfaces is doubly connected (for a single cross-cut renders each simply connected) and each of them

* *Ueber Riemann's Theorie der algebraischen Functionen und ihrer Integrale*, (Leipzig, Teubner, 1882), p. 26.

has two boundaries. If however the ribbon, previous to being closed, have undergone an odd number of twists each through 180°, the surface thus obtained is not equivalent to the single-holed portion of the plane; it is unifacial and has only one boundary.

A spherical surface pierced in $n+1$ holes is equivalent to a bounded portion of the plane with n holes; each is of connectivity $n+1$ and has $n+1$ boundaries. The spherical surface can be deformed into the plane surface by stretching one of its holes into the form of the outside boundary of the plane surface.

Ex. Prove that the surface of a bounded anchor-ring can be physically deformed into the surface in Fig. 47, p. 332.

For continuation and fuller development of the subjects of the present chapter, the following references, in addition to those which have been given, will be found useful:

Klein, *Math. Ann.*, t. vii, (1874), pp. 548—557; ib., t. ix, (1876), pp. 476—482.

Lippich, *Math. Ann.*, t. vii, (1874), pp. 212—229; *Wiener Sitzungsb.*, t. lxix, (ii), (1874), pp. 91—99.

Durège, *Wiener Sitzungsb.*, t. lxix, (ii), (1874), pp. 115—120; and section 9 of his treatise, quoted on p. 316, note.

Neumann, chapter vii of his treatise, quoted on p. 5, note.

Dyck, *Math. Ann.*, t. xxxii, (1888), pp. 457—512, ib., t. xxxvii, (1890), pp. 273—316; at the beginning of the first part of this investigation, a valuable series of references is given.

Dingeldey, *Topologische Studien*, (Leipzig, Teubner, 1890).