But (cone or segment of cone APP') : (segment APP')

$$= A'N : NH' \qquad \text{[Props. 29, 30]}$$
$$= AN . A'N : AN . NH'.$$

Therefore, *ex aequali*,

(cone or segment of cone ABB') : (segment APP')

$$= AK . CA' : AN . NH',$$

so that (spheroid) : (segment APP')

$$= HH' . AK : AN . NH',$$

since $HH' = 4CA'$.

Hence (segment $A'PP'$) : (segment APP')

$$= (HH' . AK - AN . NH') : AN . NH'$$
$$= (AK . NH + NH' . NK) : AN . NH'.$$

Further,

(segment APP') : (cone or segment of cone APP')

$$= NH' : A'N$$
$$= AN . NH' : AN . A'N,$$

and

(cone or segmt. of cone APP') : (cone or segmt. of cone $A'PP'$)

$$= AN : A'N$$
$$= AN . A'N : A'N^2.$$

From the last three proportions we obtain, *ex aequali*,

(segment $A'PP'$) : (cone or segment of cone $A'PP'$)

$$= (AK . NH + NH' . NK) : A'N^2$$
$$= (AK . NH + NH' . NK) : (CA^2 + NH' . CN)$$
$$= (AK . NH + NH' . NK) : (AK . AN + NH' . CN) \ldots (\beta).$$

But

$$AK . NH : AK . AN = NH : AN$$
$$= CA + AN : AN$$
$$= AK + CA : CA$$

(since $AK : AC = AC : AN$)

$$= HK : CA$$
$$= HK - NH : CA - AN$$
$$= NK : CN$$
$$= NH' . NK : NH' . CN.$$

Hence the ratio in (β) is equal to the ratio

$$AK . NH : AK . AN, \text{ or } NH : AN.$$

Therefore

$$(\text{segment } A'PP') : (\text{cone or segment of cone } A'PP')$$
$$= NH : AN$$
$$= CA + AN : AN.$$

[If (x, y) be the coordinates of P referred to the conjugate diameters AA', BB' as axes of x, y, and if $2a$, $2b$ be the lengths of the diameters respectively, we have, since

(spheroid) – (lesser segment) = (greater segment),

$$4 . ab^2 - \frac{2a+x}{a+x} . y^2 (a - x) = \frac{2a-x}{a-x} . y^2 (a + x);$$

and the above proposition is the geometrical proof of the truth of this equation where x, y are connected by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.]$$

ON SPIRALS.

"ARCHIMEDES to Dositheus greeting.

Of most of the theorems which I sent to Conon, and of which you ask me from time to time to send you the proofs, the demonstrations are already before you in the books brought to you by Heracleides; and some more are also contained in that which I now send you. Do not be surprised at my taking a considerable time before publishing these proofs. This has been owing to my desire to communicate them first to persons engaged in mathematical studies and anxious to investigate them. In fact, how many theorems in geometry which have seemed at first impracticable are in time successfully worked out! Now Conon died before he had sufficient time to investigate the theorems referred to; otherwise he would have discovered and made manifest all these things, and would have enriched geometry by many other discoveries besides. For I know well that it was no common ability that he brought to bear on mathematics, and that his industry was extraordinary. But, though many years have elapsed since Conon's death, I do not find that any one of the problems has been stirred by a single person. I wish now to put them in review one by one, particularly as it happens that there are two included among them which are impossible of realisation* [and which may serve as a warning] how those who claim to discover everything but produce no proofs of the same may be confuted as having actually pretended to discover the impossible.

* Heiberg reads *τέλος δὲ ποθεσόμενα*, but F has *τέλους*, so that the true reading is perhaps *τέλους δὲ ποτιδεόμενα*. The meaning appears to be simply 'wrong.'

What are the problems I mean, and what are those of which you have already received the proofs, and those of which the proofs are contained in this book respectively, I think it proper to specify. The first of the problems was, Given a sphere, to find a plane area equal to the surface of the sphere; and this was first made manifest on the publication of the book concerning the sphere, for, when it is once proved that the surface of any sphere is four times the greatest circle in the sphere, it is clear that it is possible to find a plane area equal to the surface of the sphere. The second was, Given a cone or a cylinder, to find a sphere equal to the cone or cylinder; the third, To cut a given sphere by a plane so that the segments of it have to one another an assigned ratio; the fourth, To cut a given sphere by a plane so that the segments of the surface have to one another an assigned ratio; the fifth, To make a given segment of a sphere similar to a given segment of a sphere*; the sixth, Given two segments of either the same or different spheres, to find a segment of a sphere which shall be similar to one of the segments and have its surface equal to the surface of the other segment. The seventh was, From a given sphere to cut off a segment by a plane so that the segment bears to the cone which has the same base as the segment and equal height an assigned ratio greater than that of three to two. Of all the propositions just enumerated Heracleides brought you the proofs. The proposition stated next after these was wrong, viz. that, if a sphere be cut by a plane into unequal parts, the greater segment will have to the less the duplicate ratio of that which the greater surface has to the less. That this is wrong is obvious by what I sent you before; for it included this proposition: If a sphere be cut into unequal parts by a plane at right angles to any diameter in the sphere, the greater segment of the surface will have to the less the same ratio as the greater segment of the diameter has to the less, while the greater segment of the sphere has to the less a ratio less than the duplicate ratio of that which the

* *τὸ δοθὲν τμᾶμα σφαίρας τῷ δοθέντι τμάματι σφαίρας ὁμοιῶσαι*, i.e. to make a segment of a sphere similar to one given segment and equal in content to another given segment. [Cf. *On the Sphere and Cylinder*, II. 5.]

greater surface has to the less, but greater than the sesquialterate* of that ratio. The last of the problems was also wrong, viz. that, if the diameter of any sphere be cut so that the square on the greater segment is triple of the square on the lesser segment, and if through the point thus arrived at a plane be drawn at right angles to the diameter and cutting the sphere, the figure in such a form as is the greater segment of the sphere is the greatest of all the segments which have an equal surface. That this is wrong is also clear from the theorems which I before sent you. For it was there proved that the hemisphere is the greatest of all the segments of a sphere bounded by an equal surface.

After these theorems the following were propounded concerning the cone†. If a section of a right-angled cone [a parabola], in which the diameter [axis] remains fixed, be made to revolve so that the diameter [axis] is the axis [of revolution], let the figure described by the section of the right-angled cone be called a *conoid*. And if a plane touch the conoidal figure and another plane drawn parallel to the tangent plane cut off a segment of the conoid, let the *base* of the segment cut off be defined as the cutting plane, and the *vertex* as the point in which the other plane touches the conoid. Now, if the said figure be cut by a plane at right angles to the axis, it is clear that the section will be a circle; but it needs to be proved that the segment cut off will be half as large again as the cone which has the same base as the segment and equal height. And if two segments be cut off from the conoid by planes drawn in any manner, it is clear that the sections will be sections of acute-angled cones [ellipses] if the cutting planes be not at right angles to the axis; but it needs to be proved that the segments will bear to one another the ratio of the squares on the lines drawn from their vertices parallel to the axis to meet the cutting planes. The proofs of these propositions are not yet sent to you.

After these came the following propositions about the *spiral*,

* (λόγον) *μείζονα ἢ ἡμιόλιον τοῦ, ὃν ἔχει κ.τ.λ.*, i.e. a ratio greater than (the ratio of the surfaces)$^{\frac{3}{2}}$. See *On the Sphere and Cylinder*, II. 8.

† This should be presumably 'the *conoid*,' not 'the cone.'

which are as it were another sort of problem having nothing in common with the foregoing; and I have written out the proofs of them for you in this book. They are as follows. If a straight line of which one extremity remains fixed be made to revolve at a uniform rate in a plane until it returns to the position from which it started, and if, at the same time as the straight line revolves, a point move at a uniform rate along the straight line, starting from the fixed extremity, the point will describe a spiral in the plane. I say then that the area bounded by the spiral and the straight line which has returned to the position from which it started is a third part of the circle described with the fixed point as centre and with radius the length traversed by the point along the straight line during the one revolution. And, if a straight line touch the spiral at the extreme end of the spiral, and another straight line be drawn at right angles to the line which has revolved and resumed its position from the fixed extremity of it, so as to meet the tangent, I say that the straight line so drawn to meet it is equal to the circumference of the circle. Again, if the revolving line and the point moving along it make several revolutions and return to the position from which the straight line started, I say that the area added by the spiral in the third revolution will be double of that added in the second, that in the fourth three times, that in the fifth four times, and generally the areas added in the later revolutions will be multiples of that added in the second revolution according to the successive numbers, while the area bounded by the spiral in the first revolution is a sixth part of that added in the second revolution. Also, if on the spiral described in one revolution two points be taken and straight lines be drawn joining them to the fixed extremity of the revolving line, and if two circles be drawn with the fixed point as centre and radii the lines drawn to the fixed extremity of the straight line, and the shorter of the two lines be produced, I say that (1) the area bounded by the circumference of the greater circle in the direction of (the part of) the spiral included between the straight lines, the spiral (itself) and the produced straight line will bear to (2) the area bounded by the circumference of the lesser circle, the same (part of the) spiral and the straight line joining their extremities the ratio which (3) the radius of the lesser circle together with two thirds of the excess of the radius of the greater circle over the radius of the lesser bears to (4) the radius of the lesser circle together with one third of the said excess.⌋

The proofs then of these theorems and others relating to the spiral are given in the present book. Prefixed to them, after the manner usual in other geometrical works, are the propositions necessary to the proofs of them. And here too, as in the books previously published, I assume the following lemma, that, if there be (two) unequal lines or (two) unequal areas, the excess by which the greater exceeds the less can, by being [continually] added to itself, be made to exceed any given magnitude among those which are comparable with [it and with] one another."

Proposition 1.

If a point move at a uniform rate along any line, and two lengths be taken on it, they will be proportional to the times of describing them.

Two unequal lengths are taken on a straight line, and two lengths on another straight line representing the times; and they are proved to be proportional by taking equimultiples of each length and the corresponding time after the manner of Eucl. V. Def. 5.

Proposition 2.

If each of two points on different lines respectively move along them each at a uniform rate, and if lengths be taken, one on each line, forming pairs, such that each pair are described in equal times, the lengths will be proportionals.

This is proved at once by equating the ratio of the lengths taken on one line to that of the times of description, which must also be equal to the ratio of the lengths taken on the other line.

Proposition 3.

Given any number of circles, it is possible to find a straight line greater than the sum of all their circumferences.

For we have only to describe polygons about each and then take a straight line equal to the sum of the perimeters of the polygons.

Proposition 4.

Given two unequal lines, viz. a straight line and the circumference of a circle, it is possible to find a straight line less than the greater of the two lines and greater than the less.

For, by the Lemma, the excess can, by being added a sufficient number of times to itself, be made to exceed the lesser line.

Thus e.g., if $c > l$ (where c is the circumference of the circle and l the length of the straight line), we can find a number n such that

$$n(c - l) > l.$$

Therefore $$c - l > \frac{l}{n},$$

and $$c > l + \frac{l}{n} > l.$$

Hence we have only to divide l into n equal parts and add one of them to l. The resulting line will satisfy the condition.

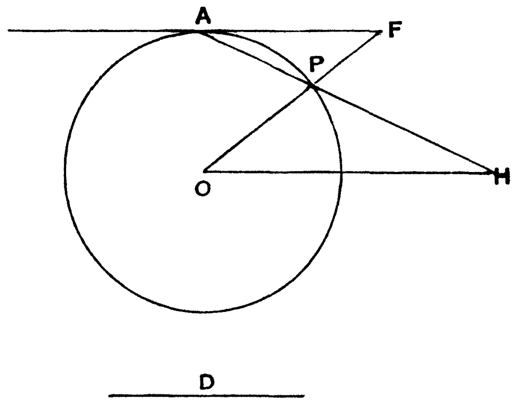
Proposition 5.

Given a circle with centre O, and the tangent to it at a point A, it is possible to draw from O a straight line OPF, meeting the circle in P and the tangent in F, such that, if c be the circumference of any given circle whatever,

$$FP : OP < (\text{arc } AP) : c.$$

Take a straight line, as D, greater than the circumference c. [Prop. 3]

Through O draw OH parallel to the given tangent, and draw through A a line APH, meeting the circle in P and OH



in H, such that the portion PH intercepted between the circle and the line OH may be equal to D*. Join OP and produce it to meet the tangent in F.

Then $$FP : OP = AP : PH, \text{ by parallels,}$$
$$= AP : D$$
$$< (\text{arc } AP) : c.$$

Proposition 6.

Given a circle with centre O, a chord AB less than the diameter, and OM the perpendicular on AB from O, it is possible to draw a straight line OFP, meeting the chord AB in F and the circle in P, such that
$$FP : PB = D : E,$$
where $D : E$ is any given ratio less than $BM : MO$.

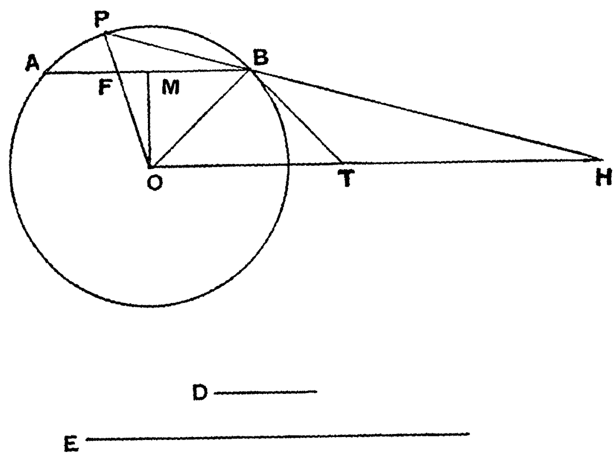
Draw OH parallel to AB, and BT perpendicular to BO meeting OH in T.

Then the triangles BMO, OBT are similar, and therefore
$$BM : MO = OB : BT,$$
whence $$D : E < OB : BT.$$

* This construction, which is assumed without any explanation as to how it is to be effected, is described in the original Greek thus: "let PH be placed (κείσθω) equal to D, verging (νεύουσα) towards A." This is the usual phraseology used in the type of problem known by the name of νεῦσις.

Suppose that a line PH (greater than BT) is taken such that

$$D : E = OB : PH,$$



and let PH be so placed that it passes through B and P lies on the circumference of the circle, while H is on the line OH*. (PH will fall outside BT, because $PH > BT$.) Join OP meeting AB in F.

We now have

$$\begin{aligned} FP : PB &= OP : PH \\ &= OB : PH \\ &= D : E. \end{aligned}$$

Proposition 7.

Given a circle with centre O, a chord AB less than the diameter, and OM the perpendicular on it from O, it is possible to draw from O a straight line OPF, meeting the circle in P and AB produced in F, such that

$$FP : PB = D : E,$$

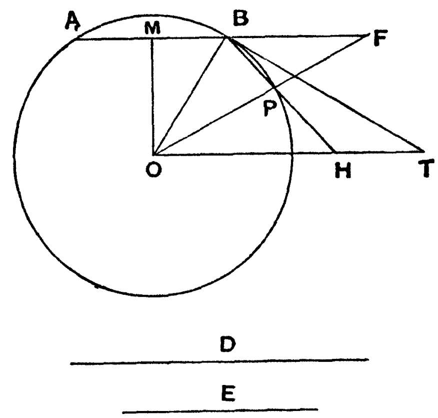
where $D : E$ is any given ratio greater than $BM : MO$.

Draw OT parallel to AB, and BT perpendicular to BO meeting OT in T.

* The Greek phrase is "let PH be placed between the circumference and the straight line (OH) through B." The construction is assumed, like the similar one in the last proposition.

In this case, $D : E > BM : MO$

$> OB : BT$, by similar triangles.



Take a line PH (less than BT) such that

$$D : E = OB : PH,$$

and place PH so that P, H are on the circle and on OT respectively, while HP produced passes through B*.

Then $FP : PB = OP : PH$

$= D : E.$

Proposition 8.

Given a circle with centre O, a chord AB less than the diameter, the tangent at B, and the perpendicular OM from O on AB, it is possible to draw from O a straight line OFP, meeting the chord AB in F, the circle in P and the tangent in G, such that

$$FP : BG = D : E,$$

where $D : E$ is any given ratio less than $BM : MO$.

If OT be drawn parallel to AB meeting the tangent at B in T,

$$BM : MO = OB : BT,$$

so that $D : E < OB : BT.$

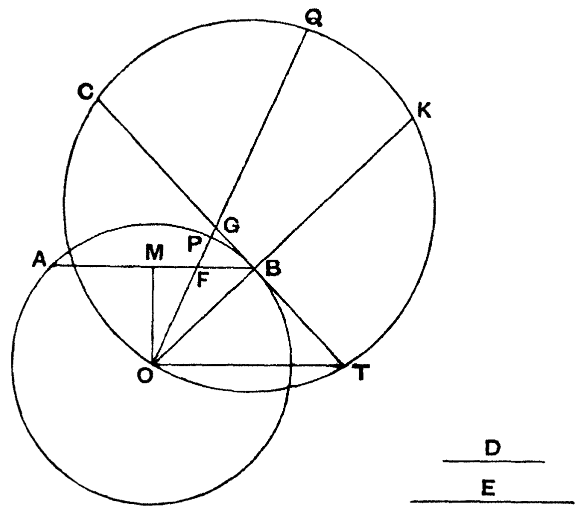
Take a point C on TB produced such that

$$D : E = OB : BC,$$

whence $BC > BT.$

* PH is described in the Greek as νεύουσαν ἐπὶ (*verging to*) the point B. As before the construction is assumed.

Through the points O, T, C describe a circle, and let OB be produced to meet this circle in K.



Then, since $BC > BT$, and OB is perpendicular to CT, it is possible to draw from O a straight line OGQ, meeting CT in G and the circle about OTC in Q, such that $GQ = BK$*.

Let OGQ meet AB in F and the original circle in P.

Now $$CG \,.\, GT = OG \,.\, GQ;$$

and $$OF : OG = BT : GT,$$

so that $$OF \,.\, GT = OG \,.\, BT.$$

It follows that

$$CG \,.\, GT : OF \,.\, GT = OG \,.\, GQ : OG \,.\, BT,$$

or
$$\begin{aligned} CG : OF &= GQ : BT \\ &= BK : BT, \text{ by construction,} \\ &= BC : OB \\ &= BC : OP. \end{aligned}$$

Hence $$OP : OF = BC : CG,$$

and therefore $$PF : OP = BG : BC,$$

or
$$\begin{aligned} PF : BG &= OP : BC \\ &= OB : BC \\ &= D : E. \end{aligned}$$

* The Greek words used are: "it is possible to place another [straight line] GQ equal to KB verging (νεύουσαν) towards O." This particular νεῦσις is discussed by Pappus (p. 298, ed. Hultsch). See the Introduction, chapter v.

Proposition 9.

Given a circle with centre O, a chord AB less than the diameter, the tangent at B, and the perpendicular OM from O on AB, it is possible to draw from O a straight line OPGF, meeting the circle in P, the tangent in G, and AB produced in F, such that

$$FP : BG = D : E,$$

where $D : E$ is any given ratio greater than $BM : MO$.

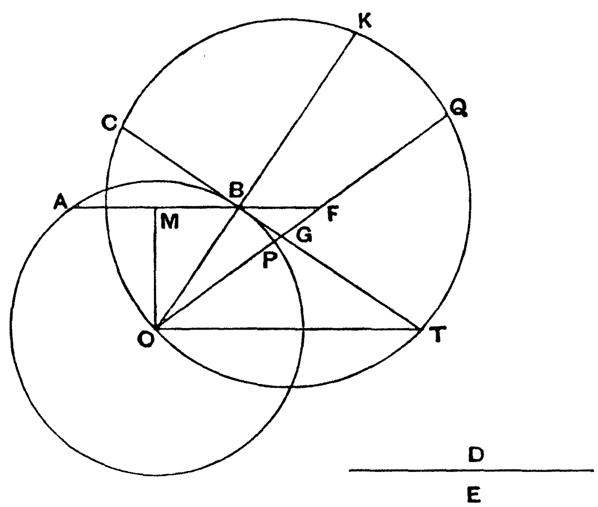
Let OT be drawn parallel to AB meeting the tangent at B in T.

Then $$D : E > BM : MO$$
$$> OB : BT, \text{ by similar triangles.}$$

Produce TB to C so that

$$D : E = OB : BC,$$

whence $$BC < BT.$$



Describe a circle through the points O, T, C, and produce OB to meet this circle in K.

Then, since $TB > BC$, and OB is perpendicular to CT, it is possible to draw from O a line OGQ, meeting CT in G, and the circle about OTC in Q, such that $GQ = BK$*. Let OQ meet the original circle in P and AB produced in F.

We now prove, exactly as in the last proposition, that

$$CG : OF = BK : BT$$
$$= BC : OP.$$

Thus, as before,

$$OP : OF = BC : CG,$$

and $$OP : PF = BC : BG,$$

whence $$PF : BG = OP : BC$$
$$= OB : BC$$
$$= D : E.$$

Proposition 10.

If $A_1, A_2, A_3, \dots A_n$ be n lines forming an ascending arithmetical progression in which the common difference is equal to A_1, the least term, then

$$(n+1)A_n^2 + A_1(A_1 + A_2 + \dots + A_n) = 3(A_1^2 + A_2^2 + \dots + A_n^2).$$

[Archimedes' proof of this proposition is given above, p. 107–9, and it is there pointed out that the result is equivalent to

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.]$$

Cor. 1. *It follows from this proposition that*

$$n \cdot A_n^2 < 3(A_1^2 + A_2^2 + \dots + A_n^2),$$

and also that

$$n \cdot A_n^2 > 3(A_1^2 + A_2^2 + \dots + A_{n-1}^2).$$

[For the proof of the latter inequality see p. 109 above.]

Cor. 2. *All the results will equally hold if similar figures are substituted for squares.*

* See the note on the last proposition.

Proposition 11.

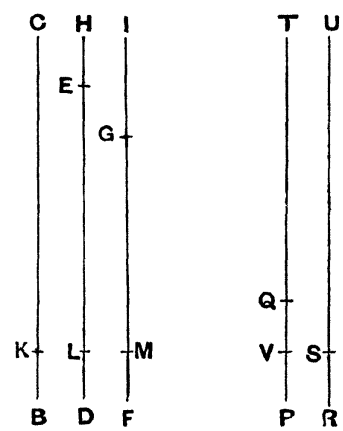
If A_1, $A_2, \ldots A_n$ be n lines forming an ascending arithmetical progression [in which the common difference is equal to the least term A_1], then*

$$(n-1)A_n^2 : (A_n^2 + A_{n-1}^2 + \ldots + A_2^2) < A_n^2 : \{A_n \cdot A_1 + \tfrac{1}{3}(A_n - A_1)^2\};$$

but

$$(n-1)A_n^2 : (A_{n-1}^2 + A_{n-2}^2 + \ldots + A_1^2) > A_n^2 : \{A_n \cdot A_1 + \tfrac{1}{3}(A_n - A_1)^2\}.$$

[Archimedes sets out the terms side by side in the manner shown in the figure, where $BC = A_n$, $DE = A_{n-1}, \ldots RS = A_1$, and produces DE, $FG, \ldots RS$ until they are respectively equal to BC or A_n, so that EH, $GI, \ldots SU$ in the figure are respectively equal to A_1, $A_2 \ldots A_{n-1}$. He further measures lengths BK, DL, $FM, \ldots PV$ along BC, DE, $FG, \ldots PQ$ respectively each equal to RS.



The figure makes the relations between the terms easier to see with the eye, but the use of so large a number of letters makes the proof somewhat difficult to follow, and it may be more clearly represented as follows.]

It is evident that $(A_n - A_1) = A_{n-1}$.

The following proportion is therefore obviously true, viz.

$$(n-1)A_n^2 : (n-1)(A_n \cdot A_1 + \tfrac{1}{3}A_{n-1}^2) = A_n^2 : \{A_n \cdot A_1 + \tfrac{1}{3}(A_n - A_1)^2\}.$$

* The proposition is true even when the common difference is not equal to A_1, and is assumed in the more general form in Props. 25 and 26. But, as Archimedes' proof assumes the equality of A_1 and the common difference, the words are here inserted to prevent misapprehension.

In order therefore to prove the desired result, we have only to show that

$$(n-1)\,A_n \,.\, A_1 + \tfrac{1}{3}(n-1)\,A_{n-1}^2 < (A_n^2 + A_{n-1}^2 + \ldots + A_2^2)$$

but $$> (A_{n-1}^2 + A_{n-2}^2 + \ldots + A_1^2).$$

I. To prove the first inequality, we have

$$\begin{aligned}(n-1)\,A_n \,.\, A_1 + \tfrac{1}{3}(n-1)\,A_{n-1}^2 \\ &= (n-1)\,A_1^2 + (n-1)\,A_1 \,.\, A_{n-1} + \tfrac{1}{3}(n-1)\,A_{n-1}^2 \ldots (1).\end{aligned}$$

And

$$\begin{aligned}A_n^2 + A_{n-1}^2 + \ldots + A_2^2 \\ &= (A_{n-1} + A_1)^2 + (A_{n-2} + A_1)^2 + \ldots + (A_1 + A_1)^2 \\ &= (A_{n-1}^2 + A_{n-2}^2 + \ldots + A_1^2) \\ &\quad + (n-1)\,A_1^2 \\ &\quad + 2A_1\,(A_{n-1} + A_{n-2} + \ldots + A_1) \\ &= (A_{n-1}^2 + A_{n-2}^2 + \ldots + A_1^2) \\ &\quad + (n-1)\,A_1^2 \\ &\quad + A_1\,\{A_{n-1} + A_{n-2} + A_{n-3} + \ldots + A_1 \\ &\qquad\quad + A_1 + A_2 + \ldots + A_{n-2} + A_{n-1}\} \\ &= (A_{n-1}^2 + A_{n-2}^2 + \ldots + A_1^2) \\ &\quad + (n-1)\,A_1^2 \\ &\quad + nA_1 \,.\, A_{n-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).\end{aligned}$$

Comparing the right-hand sides of (1) and (2), we see that $(n-1)\,A_1^2$ is common to both sides, and

$$(n-1)\,A_1 \,.\, A_{n-1} < n\,A_1 \,.\, A_{n-1},$$

while, by Prop. 10, Cor. 1,

$$\tfrac{1}{3}(n-1)\,A_{n-1}^2 < A_{n-1}^2 + A_{n-2}^2 + \ldots + A_1^2.$$

It follows therefore that

$$(n-1)\,A_n \,.\, A_1 + \tfrac{1}{3}(n-1)\,A_{n-1}^2 < (A_n^2 + A_{n-1}^2 + \ldots + A_2^2);$$

and hence the first part of the proposition is proved.

II. We have now, in order to prove the second result, to show that

$$(n-1)\,A_n \,.\, A_1 + \tfrac{1}{3}(n-1)\,A_{n-1}^2 > (A_{n-1}^2 + A_{n-2}^2 + \ldots + A_1^2).$$

The right-hand side is equal to

$$\begin{aligned}(A_{n-2}+A_1)^2+(A_{n-3}+A_1)^2+\ldots+(A_1+A_1)^2+A_1^2 \\ &= A_{n-2}^2+A_{n-3}^2+\ldots+A_1^2 \\ &\quad +(n-1)A_1^2 \\ &\quad +2A_1(A_{n-2}+A_{n-3}+\ldots+A_1) \\ &= (A_{n-2}^2+A_{n-3}^2+\ldots+A_1^2) \\ &\quad +(n-1)A_1^2 \\ &\quad +A_1\left\{\begin{matrix} A_{n-2}+A_{n-3}+\ldots+A_1 \\ +A_1+A_2+\ldots+A_{n-2}\end{matrix}\right\} \\ &= (A_{n-2}^2+A_{n-3}^2+\ldots+A_1^2) \\ &\quad +(n-1)A_1^2 \\ &\quad +(n-2)A_1 \, . \, A_{n-1} \ldots\ldots\ldots\ldots(3).\end{aligned}$$

Comparing this expression with the right-hand side of (1) above, we see that $(n-1)A_1^2$ is common to both sides, and

$$(n-1)A_1 \, . \, A_{n-1} > (n-2)A_1 \, . \, A_{n-1},$$

while, by Prop. 10, Cor. 1,

$$\tfrac{1}{3}(n-1)A_{n-1}^2 > (A_{n-2}^2+A_{n-3}^2+\ldots+A_1^2).$$

Hence

$$(n-1)A_n \, . \, A_1+\tfrac{1}{3}(n-1)A_{n-1}^2 > (A_{n-1}^2+A_{n-2}^2+\ldots+A_1^2);$$

and the second required result follows.

COR. *The results in the above proposition are equally true if similar figures be substituted for squares on the several lines.*

DEFINITIONS.

1. If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves, a point move at a uniform rate along the straight line beginning from the extremity which remains fixed, the point will describe a **spiral** (ἕλιξ) in the plane.

2. Let the extremity of the straight line which remains

fixed while the straight line revolves be called the **origin*** (ἀρχά) of the spiral.

3. And let the position of the line from which the straight line began to revolve be called the **initial line*** in the revolution (ἀρχὰ τᾶς περιφορᾶς).

4. Let the length which the point that moves along the straight line describes in one revolution be called the **first distance**, that which the same point describes in the second revolution the **second distance**, and similarly let the distances described in further revolutions be called after the number of the particular revolution.

5. Let the area bounded by the spiral described in the first revolution and the *first distance* be called the **first area**, that bounded by the spiral described in the second revolution and the *second distance* the **second area**, and similarly for the rest in order.

6. If from the origin of the spiral any straight line be drawn, let that side of it which is in the same direction as that of the revolution be called **forward** (προαγούμενα), and that which is in the other direction **backward** (ἑπόμενα).

7. Let the circle drawn with the *origin* as centre and the *first distance* as radius be called the **first circle**, that drawn with the same centre and twice the radius the **second circle**, and similarly for the succeeding circles.

Proposition 12.

If any number of straight lines drawn from the origin to meet the spiral make equal angles with one another, the lines will be in arithmetical progression.

[The proof is obvious.]

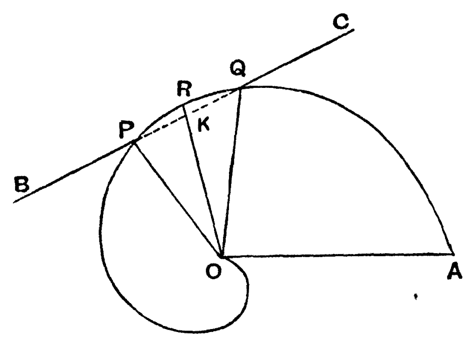
* The literal translation would of course be the "beginning of the spiral" and "the beginning of the revolution" respectively. But the modern names will be more suitable for use later on, and are therefore employed here.

Proposition 13.

If a straight line touch the spiral, it will touch it in one point only.

Let O be the origin of the spiral, and BC a tangent to it.

If possible, let BC touch the spiral in two points P, Q. Join OP, OQ, and bisect the angle POQ by the straight line OR meeting the spiral in R.



Then [Prop. 12] OR is an arithmetic mean between OP and OQ, or

$$OP + OQ = 2OR.$$

But in any triangle POQ, if the bisector of the angle POQ meets PQ in K,

$$OP + OQ > 2OK^*.$$

Therefore $OK < OR$, and it follows that some point on BC between P and Q lies within the spiral. Hence BC cuts the spiral; which is contrary to the hypothesis.

Proposition 14.

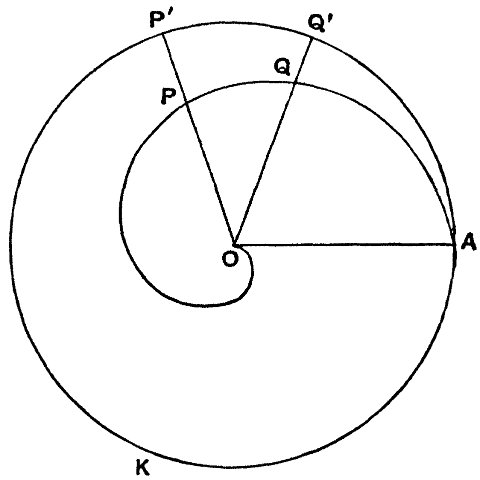
If O be the origin, and P, Q two points on the first turn of the spiral, and if OP, OQ produced meet the 'first circle' $AKP'Q'$ in P', Q' respectively, OA being the initial line, then

$$OP : OQ = (arc\ AKP') : (arc\ AKQ').$$

For, while the revolving line OA moves about O, the point A on it moves uniformly along the circumference of the circle

* This is assumed as a known proposition; but it is easily proved.

$AKP'Q'$, and at the same time the point describing the spiral moves uniformly along OA.



Thus, while A describes the arc AKP', the moving point on OA describes the length OP, and, while A describes the arc AKQ', the moving point on OA describes the distance OQ.

Hence $OP : OQ = (\text{arc } AKP') : (\text{arc } AKQ')$. [Prop. 2]

Proposition 15.

If P, Q be points on the second turn of the spiral, and OP, OQ meet the 'first circle' $AKP'Q'$ in P', Q', as in the last proposition, and if c be the circumference of the first circle, then

$$OP : OQ = c + (\text{arc } AKP') : c + (\text{arc } AKQ').$$

For, while the moving point on OA describes the distance OP, the point A describes the whole of the circumference of the 'first circle' together with the arc AKP'; and, while the moving point on OA describes the distance OQ, the point A describes the whole circumference of the 'first circle' together with the arc AKQ'.

COR. Similarly, if P, Q are on the nth turn of the spiral,

$$OP : OQ = (n-1)\,c + (\text{arc } AKP') : (n-1)\,c + (\text{arc } AKQ').$$