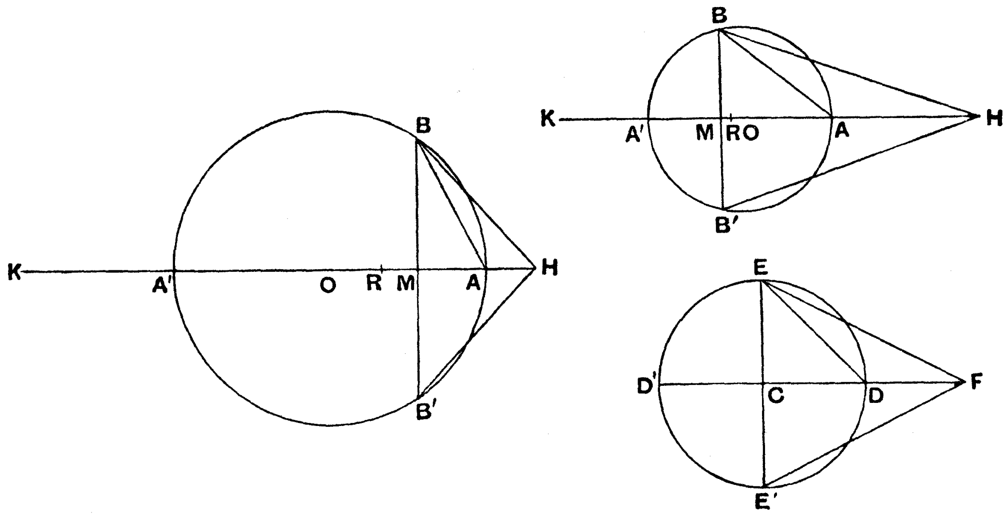
Suppose the surfaces of the segment $ABB'$ and of the hemisphere $DEE'$ to be equal.




Since the surfaces are equal, $AB = DE$. [I. 42, 43]

Now, in Fig. 1, $AB^2 > 2AM^2$ and $< 2AO^2$,

and, in Fig. 2, $AB^2 < 2AM^2$ and $> 2AO^2$.

Hence, if $R$ be taken on $AA'$ such that

$$AR^2 = \tfrac{1}{2}AB^2,$$

$R$ will fall between $O$ and $M$.

Also, since $AB^2 = DE^2$, $AR = CD$.

Produce $OA'$ to $K$ so that $OA' = A'K$, and produce $A'A$ to $H$ so that

$$A'K : A'M = HA : AM,$$

or, *componendo*, $A'K + A'M : A'M = HM : MA$ ............(1).

Thus the cone $HBB'$ is equal to the segment $ABB'$. [Prop. 2]

Again, produce $CD$ to $F$ so that $CD = DF$, and the cone $FEE'$ will be equal to the hemisphere $DEE'$. [Prop. 2]

Now $AR . RA' > AM . MA'$,

and $AR^2 = \tfrac{1}{2}AB^2 = \tfrac{1}{2}AM . AA' = AM . A'K.$

Hence

$$AR \,.\, RA' + RA^2 > AM \,.\, MA' + AM \,.\, A'K,$$

or

$$AA' \,.\, AR > AM \,.\, MK$$

$$> HM \,.\, A'M, \text{ by (1).}$$

Therefore $\quad AA' : A'M > HM : AR,$

or $\quad AB^2 : BM^2 > HM : AR,$

i.e. $\quad AR^2 : BM^2 > HM : 2AR$, since $AB^2 = 2AR^2$,

$$> HM : CF.$$

Thus, since $AR = CD$, or $CE$,

(circle on diam. $EE'$) : (circle on diam. $BB'$) $> HM : CF$.

It follows that

(the cone $FEE'$) $>$ (the cone $HBB'$),

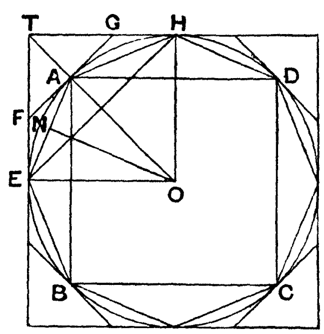
and therefore the hemisphere $DEE'$ is greater in volume than the segment $ABB'$.

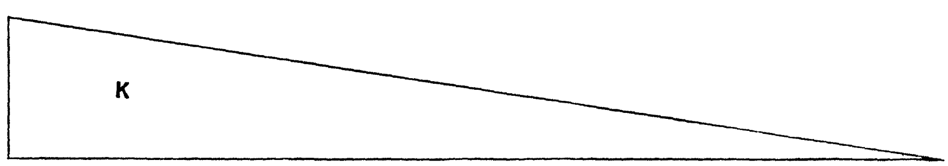
# MEASUREMENT OF A CIRCLE.

## Proposition 1.

*The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.*

Let $ABCD$ be the given circle, $K$ the triangle described.




Then, if the circle is not equal to $K$, it must be either greater or less.

I. If possible, let the circle be greater than $K$.

Inscribe a square $ABCD$, bisect the arcs $AB$, $BC$, $CD$, $DA$, then bisect (if necessary) the halves, and so on, until the sides of the inscribed polygon whose angular points are the points of division subtend segments whose sum is less than the excess of the area of the circle over $K$.

Thus the area of the polygon is greater than $K$.

Let $AE$ be any side of it, and $ON$ the perpendicular on $AE$ from the centre $O$.

Then $ON$ is less than the radius of the circle and therefore less than one of the sides about the right angle in $K$. Also the perimeter of the polygon is less than the circumference of the circle, i.e. less than the other side about the right angle in $K$.

Therefore the area of the polygon is less than $K$; which is inconsistent with the hypothesis.

Thus the area of the circle is not greater than $K$.

II. If possible, let the circle be less than $K$.

Circumscribe a square, and let two adjacent sides, touching the circle in $E$, $H$, meet in $T$. Bisect the arcs between adjacent points of contact and draw the tangents at the points of bisection. Let $A$ be the middle point of the arc $EH$, and $FAG$ the tangent at $A$.

Then the angle $TAG$ is a right angle.

$$\text{Therefore} \qquad TG > GA$$
$$> GH.$$

It follows that the triangle $FTG$ is greater than half the area $TEAH$.

Similarly, if the arc $AH$ be bisected and the tangent at the point of bisection be drawn, it will cut off from the area $GAH$ more than one-half.

Thus, by continuing the process, we shall ultimately arrive at a circumscribed polygon such that the spaces intercepted between it and the circle are together less than the excess of $K$ over the area of the circle.

Thus the area of the polygon will be less than $K$.

Now, since the perpendicular from $O$ on any side of the polygon is equal to the radius of the circle, while the perimeter of the polygon is greater than the circumference of the circle, it follows that the area of the polygon is greater than the triangle $K$; which is impossible.

Therefore the area of the circle is not less than $K$.

Since then the area of the circle is neither greater nor less than $K$, it is equal to it.

## Proposition 2.

*The area of a circle is to the square on its diameter as* 11 *to* 14.

[The text of this proposition is not satisfactory, and Archimedes cannot have placed it before Proposition 3, as the approximation depends upon the result of that proposition.]

## Proposition 3.

*The ratio of the circumference of any circle to its diameter is less than* $3\frac{1}{7}$ *but greater than* $3\frac{10}{71}$.

[In view of the interesting questions arising out of the arithmetical content of this proposition of Archimedes, it is necessary, in reproducing it, to distinguish carefully the actual steps set out in the text as we have it from the intermediate steps (mostly supplied by Eutocius) which it is convenient to put in for the purpose of making the proof easier to follow. Accordingly all the steps not actually appearing in the text have been enclosed in square brackets, in order that it may be clearly seen how far Archimedes omits actual calculations and only gives results. It will be observed that he gives two fractional approximations to $\sqrt{3}$ (one being less and the other greater than the real value) without any explanation as to how he arrived at them; and in like manner approximations to the square roots of several large numbers which are not complete squares are merely stated. These various approximations and the machinery of Greek arithmetic in general will be found discussed in the Introduction, Chapter IV.]

I. Let $AB$ be the diameter of any circle, $O$ its centre, $AC$ the tangent at $A$; and let the angle $AOC$ be one-third of a right angle.

Then $OA : AC\,[=\sqrt{3} : 1] > 265 : 153$ ............ (1),

and $OC : CA\,[=2 : 1] = 306 : 153$ .............. (2).

*First,* draw $OD$ bisecting the angle $AOC$ and meeting $AC$ in $D$.

Now $CO : OA = CD : DA,$ [Eucl. VI. 3]

so that $[CO + OA : OA = CA : DA,$ or]

$$CO + OA : CA = OA : AD.$$

Therefore [by (1) and (2)]

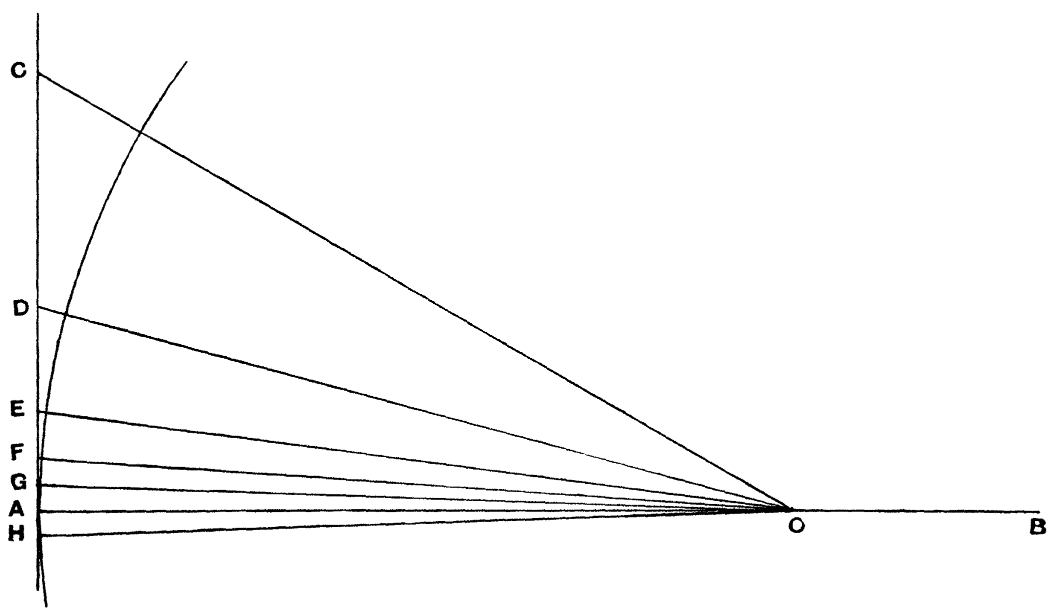
$$OA : AD > 571 : 153 \quad\text{.................. (3).}$$

Hence $OD^2 : AD^2\,[=(OA^2 + AD^2) : AD^2$

$$> (571^2 + 153^2) : 153^2]$$

$$> 349450 : 23409,$$

so that $OD : DA > 591\frac{1}{8} : 153$ .................... (4).




*Secondly,* let $OE$ bisect the angle $AOD$, meeting $AD$ in $E$.

[Then $DO : OA = DE : EA,$

so that $DO + OA : DA = OA : AE.$]

Therefore $OA : AE\,[> (591\frac{1}{8} + 571) : 153,$ by (3) and (4)]

$$> 1162\frac{1}{8} : 153 \quad\text{.................... (5).}$$

[It follows that

$$OE^2 : EA^2 > \{(1162\tfrac{1}{8})^2 + 153^2\} : 153^2$$
$$> (1350534\tfrac{33}{64} + 23409) : 23409$$
$$> 1373943\tfrac{33}{64} : 23409.]$$

Thus $$OE : EA > 1172\tfrac{1}{8} : 153 \qquad (6).$$

*Thirdly*, let $OF$ bisect the angle $AOE$ and meet $AE$ in $F$.

We thus obtain the result [corresponding to (3) and (5) above] that

$$OA : AF [> (1162\tfrac{1}{8} + 1172\tfrac{1}{8}) : 153]$$
$$> 2334\tfrac{1}{4} : 153 \qquad (7).$$

[Therefore $$OF^2 : FA^2 > \{(2334\tfrac{1}{4})^2 + 153^2\} : 153^2$$
$$> 5472132\tfrac{1}{16} : 23409.]$$

Thus $$OF : FA > 2339\tfrac{1}{4} : 153 \qquad (8).$$

*Fourthly*, let $OG$ bisect the angle $AOF$, meeting $AF$ in $G$.

We have then

$$OA : AG [> (2334\tfrac{1}{4} + 2339\tfrac{1}{4}) : 153, \text{ by means of (7) and (8)}]$$
$$> 4673\tfrac{1}{2} : 153.$$

Now the angle $AOC$, which is one-third of a right angle, has been bisected four times, and it follows that

$$\angle AOG = \tfrac{1}{48} \text{ (a right angle)}.$$

Make the angle $AOH$ on the other side of $OA$ equal to the angle $AOG$, and let $GA$ produced meet $OH$ in $H$.

Then $$\angle GOH = \tfrac{1}{24} \text{ (a right angle)}.$$

Thus $GH$ is one side of a regular polygon of 96 sides circumscribed to the given circle.

And, since $$OA : AG > 4673\tfrac{1}{2} : 153,$$
while $$AB = 2OA, \quad GH = 2AG,$$
it follows that

$$AB : (\text{perimeter of polygon of 96 sides}) [> 4673\tfrac{1}{2} : 153 \times 96]$$
$$> 4673\tfrac{1}{2} : 14688.$$

But
$$\frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}}$$
$$\left[< 3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}}\right]$$
$$< 3\tfrac{1}{7}.$$

Therefore the circumference of the circle (being less than the perimeter of the polygon) is *a fortiori* less than $3\frac{1}{7}$ times the diameter $AB$.

II. Next let $AB$ be the diameter of a circle, and let $AC$, meeting the circle in $C$, make the angle $CAB$ equal to one-third of a right angle. Join $BC$.

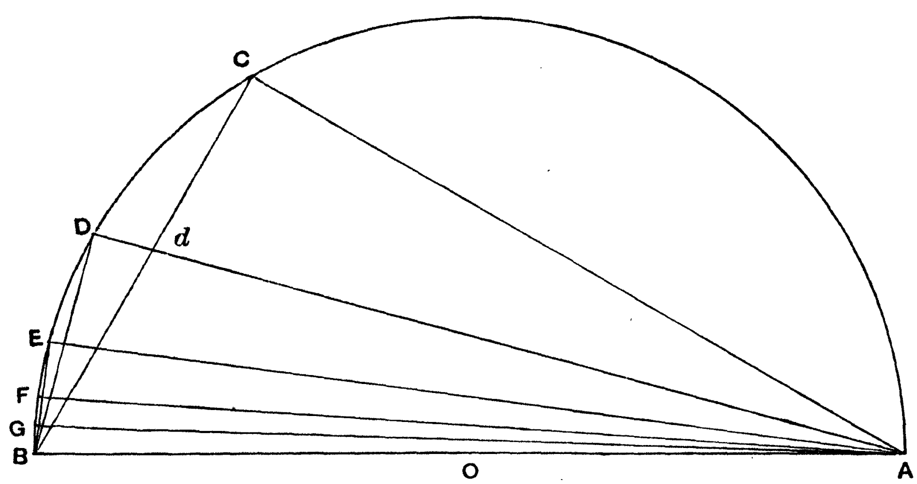
Then $\qquad AC : CB\,[=\sqrt{3} : 1] < 1351 : 780.$

*First*, let $AD$ bisect the angle $BAC$ and meet $BC$ in $d$ and the circle in $D$. Join $BD$.

Then
$$\angle BAD = \angle dAC$$
$$= \angle dBD,$$

and the angles at $D$, $C$ are both right angles.

It follows that the triangles $ADB$, $[ACd]$, $BDd$ are similar.




Therefore
$$AD : DB = BD : Dd$$
$$[= AC : Cd]$$
$$= AB : Bd \qquad \text{[Eucl. VI. 3]}$$
$$= AB + AC : Bd + Cd$$
$$= AB + AC : BC$$

or
$$BA + AC : BC = AD : DB.$$

[But $\quad AC : CB < 1351 : 780$, from above,

while $\quad BA : BC = 2 : 1$

$$= 1560 : 780.]$$

Therefore $\quad AD : DB < 2911 : 780$ ..................(1).

[Hence $\quad AB^2 : BD^2 < (2911^2 + 780^2) : 780^2$

$$< 9082321 : 608400.]$$

Thus $\quad AB : BD < 3013\tfrac{3}{4} : 780$ .............. (2).

*Secondly*, let $AE$ bisect the angle $BAD$, meeting the circle in $E$; and let $BE$ be joined.

Then we prove, in the same way as before, that

$$AE : EB [= BA + AD : BD$$

$$< (3013\tfrac{3}{4} + 2911) : 780, \text{ by (1) and (2)}]$$

$$< 5924\tfrac{3}{4} : 780$$

$$< 5924\tfrac{3}{4} \times \tfrac{4}{13} : 780 \times \tfrac{4}{13}$$

$$< 1823 : 240 \text{ .......................... (3).}$$

[Hence $\quad AB^2 : BE^2 < (1823^2 + 240^2) : 240^2$

$$< 3380929 : 57600.]$$

Therefore $\quad AB : BE < 1838\tfrac{9}{11} : 240$ ..................(4).

*Thirdly*, let $AF$ bisect the angle $BAE$, meeting the circle in $F$.

Thus $\quad AF : FB [= BA + AE : BE$

$$< 3661\tfrac{9}{11} : 240, \text{ by (3) and (4)}]$$

$$< 3661\tfrac{9}{11} \times \tfrac{11}{40} : 240 \times \tfrac{11}{40}$$

$$< 1007 : 66 \text{ ......................(5).}$$

[It follows that

$$AB^2 : BF^2 < (1007^2 + 66^2) : 66^2$$

$$< 1018405 : 4356.]$$

Therefore $\quad AB : BF < 1009\tfrac{1}{6} : 66$ ....................(6).

*Fourthly*, let the angle $BAF$ be bisected by $AG$ meeting the circle in $G$.

Then $\quad AG : GB [= BA + AF : BF]$

$$< 2016\tfrac{1}{6} : 66, \text{ by (5) and (6).}$$

[And $$AB^2 : BG^2 < \{(2016\tfrac{1}{6})^2 + 66^2\} : 66^2$$
$$< 4069284\tfrac{1}{36} : 4356.]$$

Therefore $$AB : BG < 2017\tfrac{1}{4} : 66,$$

whence $$BG : AB > 66 : 2017\tfrac{1}{4} \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

[Now the angle $BAG$ which is the result of the fourth bisection of the angle $BAC$, or of one-third of a right angle, is equal to one-fortyeighth of a right angle.

Thus the angle subtended by $BG$ at the centre is

$$\tfrac{1}{24} \text{ (a right angle).]}$$

Therefore $BG$ is a side of a regular inscribed polygon of 96 sides.

It follows from (7) that

$$\text{(perimeter of polygon)} : AB\ [> 96 \times 66 : 2017\tfrac{1}{4}]$$
$$> 6336 : 2017\tfrac{1}{4}.$$

And $$\frac{6336}{2017\frac{1}{4}} > 3\tfrac{10}{71}.$$

Much more then is the circumference of the circle greater than $3\frac{10}{71}$ times the diameter.

Thus the ratio of the circumference to the diameter

$$< 3\tfrac{1}{7} \text{ but } > 3\tfrac{10}{71}.$$

# ON CONOIDS AND SPHEROIDS.

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### Introduction*.

"ARCHIMEDES to Dositheus greeting.

In this book I have set forth and send you the proofs of the remaining theorems not included in what I sent you before, and also of some others discovered later which, though I had often tried to investigate them previously, I had failed to arrive at because I found their discovery attended with some difficulty. And this is why even the propositions themselves were not published with the rest. But afterwards, when I had studied them with greater care, I discovered what I had failed in before.

Now the remainder of the earlier theorems were propositions concerning the right-angled conoid [paraboloid of revolution]; but the discoveries which I have now added relate to an obtuse-angled conoid [hyperboloid of revolution] and to spheroidal figures, some of which I call *oblong* (*παραμάκεα*) and others *flat* (*ἐπιπλατέα*).

I. Concerning the *right-angled conoid* it was laid down that, if a section of a right-angled cone [a parabola] be made to revolve about the diameter [axis] which remains fixed and

* The whole of this introductory matter, including the definitions, is translated literally from the Greek text in order that the terminology of Archimedes may be faithfully represented. When this has once been set out, nothing will be lost by returning to modern phraseology and notation. These will accordingly be employed, as usual, when we come to the actual propositions of the treatise.

return to the position from which it started, the figure comprehended by the section of the right-angled cone is called a **right-angled conoid,** and the diameter which has remained fixed is called its **axis,** while its **vertex** is the point in which the axis meets (ἅπτεται) the surface of the conoid. And if a plane touch the right-angled conoid, and another plane drawn parallel to the tangent plane cut off a segment of the conoid, the **base** of the segment cut off is defined as the portion intercepted by the section of the conoid on the cutting plane, the **vertex** [of the segment] as the point in which the first plane touches the conoid, and the **axis** [of the segment] as the portion cut off within the segment from the line drawn through the vertex of the segment parallel to the axis of the conoid.

The questions propounded for consideration were

(1) why, if a segment of the right-angled conoid be cut off by a plane at right angles to the axis, will the segment so cut off be half as large again as the cone which has the same base as the segment and the same axis, and

(2) why, if two segments be cut off from the right-angled conoid by planes drawn in any manner, will the segments so cut off have to one another the duplicate ratio of their axes.

II. Respecting the *obtuse-angled conoid* we lay down the following premisses. If there be in a plane a section of an obtuse-angled cone [a hyperbola], its diameter [axis], and the nearest lines to the section of the obtuse-angled cone [*i.e.* the asymptotes of the hyperbola], and if, the diameter [axis] remaining fixed, the plane containing the aforesaid lines be made to revolve about it and return to the position from which it started, the nearest lines to the section of the obtuse-angled cone [the asymptotes] will clearly comprehend an isosceles cone whose vertex will be the point of concourse of the nearest lines and whose axis will be the diameter [axis] which has remained fixed. The figure comprehended by the section of the obtuse-angled cone is called an **obtuse-angled conoid** [hyperboloid of revolution], its **axis** is the diameter which has remained fixed, and its **vertex** the point in which the axis meets the surface of the conoid. The cone comprehended by the nearest lines to the section of the obtuse-angled cone is called [the cone] **enveloping the conoid** (*περιέχων τὸ κωνοειδές*), and the straight line between the vertex of the conoid and the vertex of the cone enveloping the conoid is called [the line] **adjacent to the axis** (*ποτεοῦσα τῷ ἄξονι*). And if a plane touch the obtuse-angled conoid, and another plane drawn parallel to the tangent plane cut off a segment of the conoid, the **base** of the segment so cut off is defined as the portion intercepted by the section of the conoid on the cutting plane, the **vertex** [of the segment] as the point of contact of the plane which touches the conoid, the **axis** [of the segment] as the portion cut off within the segment from the line drawn through the vertex of the segment and the vertex of the cone enveloping the conoid; and the straight line between the said vertices is called **adjacent to the axis.**

Right-angled conoids are all similar; but of obtuse-angled conoids let those be called similar in which the cones enveloping the conoids are similar.

The following questions are propounded for consideration,

(1) why, if a segment be cut off from the obtuse-angled conoid by a plane at right angles to the axis, the segment so cut off has to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of the axis of the segment and three times the line adjacent to the axis bears to the line equal to the sum of the axis of the segment and twice the line adjacent to the axis, and

(2) why, if a segment of the obtuse-angled conoid be cut off by a plane not at right angles to the axis, the segment so cut off will bear to the figure which has the same base as the segment and the same axis, being a segment of a cone* (*ἀπότμαμα κώνου*), the ratio which the line equal to the sum of the axis of the segment and three times the line adjacent to the axis bears to the line equal to the sum of the axis of the segment and twice the line adjacent to the axis.

* A *segment of a cone* is defined later (p. 104).

III. Concerning spheroidal figures we lay down the following premisses. If a section of an acute-angled cone [ellipse] be made to revolve about the greater diameter [major axis] which remains fixed and return to the position from which it started, the figure comprehended by the section of the acute-angled cone is called an **oblong spheroid** (παραμάκες σφαιροειδές). But if the section of the acute-angled cone revolve about the lesser diameter [minor axis] which remains fixed and return to the position from which it started, the figure comprehended by the section of the acute-angled cone is called a **flat spheroid** (ἐπιπλατὺ σφαιροειδές). In either of the spheroids the **axis** is defined as the diameter [axis] which has remained fixed, the **vertex** as the point in which the axis meets the surface of the spheroid, the **centre** as the middle point of the axis, and the **diameter** as the line drawn through the centre at right angles to the axis. And, if parallel planes touch, without cutting, either of the spheroidal figures, and if another plane be drawn parallel to the tangent planes and cutting the spheroid, the **base** of the resulting segments is defined as the portion intercepted by the section of the spheroid on the cutting plane, their **vertices** as the points in which the parallel planes touch the spheroid, and their **axes** as the portions cut off within the segments from the straight line joining their vertices. And that the planes touching the spheroid meet its surface at one point only, and that the straight line joining the points of contact passes through the centre of the spheroid, we shall prove. Those spheroidal figures are called **similar** in which the axes have the same ratio to the 'diameters.' And let segments of spheroidal figures and conoids be called **similar** if they are cut off from similar figures and have their bases similar, while their axes, being either at right angles to the planes of the bases or making equal angles with the corresponding diameters [axes] of the bases, have the same ratio to one another as the corresponding diameters [axes] of the bases.

The following questions about spheroids are propounded for consideration,

(1) why, if one of the spheroidal figures be cut by a plane

through the centre at right angles to the axis, each of the resulting segments will be double of the cone having the same base as the segment and the same axis; while, if the plane of section be at right angles to the axis without passing through the centre, (*a*) the greater of the resulting segments will bear to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the straight line which is the axis of the spheroid and the axis of the lesser segment bears to the axis of the lesser segment, and (*b*) the lesser segment bears to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the straight line which is the axis of the spheroid and the axis of the greater segment bears to the axis of the greater segment;

(2) why, if one of the spheroids be cut by a plane passing through the centre but not at right angles to the axis, each of the resulting segments will be double of the figure having the same base as the segment and the same axis and consisting of a segment of a cone*.

(3) But, if the plane cutting the spheroid be neither through the centre nor at right angles to the axis, (*a*) the greater of the resulting segments will have to the figure which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the line joining the vertices of the segments and the axis of the lesser segment bears to the axis of the lesser segment, and (*b*) the lesser segment will have to the figure with the same base as the segment and the same axis the ratio which the line equal to the sum of half the line joining the vertices of the segments and the axis of the greater segment bears to the axis of the greater segment. And the figure referred to is in these cases also a segment of a cone*.

When the aforesaid theorems are proved, there are discovered by means of them many theorems and problems.

Such, for example, are the theorems

(1) that similar spheroids and similar segments both of

* See the definition of a *segment of a cone* (ἀπότμαμα κώνου) on p. 104.

spheroidal figures and conoids have to one another the triplicate ratio of their axes, and

(2) that in equal spheroidal figures the squares on the 'diameters' are reciprocally proportional to the axes, and, if in spheroidal figures the squares on the 'diameters' are reciprocally proportional to the axes, the spheroids are equal.

Such also is the problem, From a given spheroidal figure or conoid to cut off a segment by a plane drawn parallel to a given plane so that the segment cut off is equal to a given cone or cylinder or to a given sphere.

After prefixing therefore the theorems and directions (ἐπιτάγματα) which are necessary for the proof of them, I will then proceed to expound the propositions themselves to you. Farewell.

DEFINITIONS.

If a cone be cut by a plane meeting all the sides [generators] of the cone, the section will be either a circle or a section of an acute-angled cone [an ellipse]. If then the section be a circle, it is clear that the segment cut off from the cone towards the same parts as the vertex of the cone will be a cone. But, if the section be a section of an acute-angled cone [an ellipse], let the figure cut off from the cone towards the same parts as the vertex of the cone be called a **segment of a cone.** Let the **base** of the segment be defined as the plane comprehended by the section of the acute-angled cone, its **vertex** as the point which is also the vertex of the cone, and its **axis** as the straight line joining the vertex of the cone to the centre of the section of the acute-angled cone.

And if a cylinder be cut by two parallel planes meeting all the sides [generators] of the cylinder, the sections will be either circles or sections of acute-angled cones [ellipses] equal and similar to one another. If then the sections be circles, it is clear that the figure cut off from the cylinder between the parallel planes will be a cylinder. But, if the sections be sections of acute-angled cones [ellipses], let the figure cut off from the cylinder between the parallel planes be called a **frustum** (τόμος) **of a cylinder.** And let the **bases** of the frustum be defined as the planes comprehended by the sections of the acute-angled cones [ellipses], and the **axis** as the straight line joining the centres of the sections of the acute-angled cones, so that the axis will be in the same straight line with the axis of the cylinder."

### Lemma.

*If in an ascending arithmetical progression consisting of the magnitudes $A_1, A_2, \dots A_n$ the common difference be equal to the least term $A_1$, then*

$$n \cdot A_n < 2(A_1 + A_2 + \dots + A_n),$$

*and*

$$> 2(A_1 + A_2 + \dots + A_{n-1}).$$

[The proof of this is given incidentally in the treatise *On Spirals*, Prop. 11. By placing lines side by side to represent the terms of the progression and then producing each so as to make it equal to the greatest term, Archimedes gives the equivalent of the following proof.

If $$S_n = A_1 + A_2 + \dots + A_{n-1} + A_n,$$

we have also $$S_n = A_n + A_{n-1} + A_{n-2} + \dots + A_1.$$

And $$A_1 + A_{n-1} = A_2 + A_{n-2} = \dots = A_n.$$

Therefore $$2S_n = (n+1) A_n,$$

whence $$n \cdot A_n < 2S_n,$$

and $$n \cdot A_n > 2S_{n-1}.$$

Thus, if the progression is $a, 2a, \dots na$,

$$S_n = \frac{n(n+1)}{2} a,$$

and $$n^2 a < 2S_n,$$

but $$> 2S_{n-1}.]$$

### Proposition 1.

*If $A_1, B_1, C_1, \dots K_1$ and $A_2, B_2, C_2, \dots K_2$ be two series of magnitudes such that*

$$\left.\begin{array}{l} A_1 : B_1 = A_2 : B_2, \\ B_1 : C_1 = B_2 : C_2, \text{ and so on} \end{array}\right\} \dots\dots\dots\dots\dots (\alpha),$$

*and if* $A_3, B_3, C_3, \ldots K_3$ *and* $A_4, B_4, C_4, \ldots K_4$ *be two other series such that*

$$\left.\begin{array}{l} A_1 : A_3 = A_2 : A_4, \\ B_1 : B_3 = B_2 : B_4, \text{ and so on} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (\beta),$$

*then* $(A_1 + B_1 + C_1 + \ldots + K_1) : (A_3 + B_3 + C_3 + \ldots + K_3)$

$$= (A_2 + B_2 + C_2 + \ldots + K_2) : (A_4 + B_4 + \ldots + K_4).$$

The proof is as follows.

Since $A_3 : A_1 = A_4 : A_2$,

and $A_1 : B_1 = A_2 : B_2$,

while $B_1 : B_3 = B_2 : B_4$,

$$\left.\begin{array}{ll} \text{we have, } \textit{ex aequali}, & A_3 : B_3 = A_4 : B_4. \\ \quad\text{Similarly} & B_3 : C_3 = B_4 : C_4, \text{ and so on} \end{array}\right\} \ldots\ldots (\gamma).$$

Again, it follows from equations ($\alpha$) that

$$A_1 : A_2 = B_1 : B_2 = C_1 : C_2 = \ldots.$$

Therefore

$$A_1 : A_2 = (A_1 + B_1 + C_1 + \ldots + K_1) : (A_2 + B_2 + \ldots + K_2),$$

or $(A_1 + B_1 + C_1 + \ldots + K_1) : A_1 = (A_2 + B_2 + C_2 + \ldots + K_2) : A_2$;

and $A_1 : A_3 = A_2 : A_4$,

while from equations ($\gamma$) it follows in like manner that

$$A_3 : (A_3 + B_3 + C_3 + \ldots + K_3) = A_4 : (A_4 + B_4 + C_4 + \ldots + K_4).$$

By the last three equations, *ex aequali*,

$$(A_1 + B_1 + C_1 + \ldots + K_1) : (A_3 + B_3 + C_3 + \ldots + K_3)$$
$$= (A_2 + B_2 + C_2 + \ldots + K_2) : (A_4 + B_4 + C_4 + \ldots + K_4).$$

Cor. If any terms in the third and fourth series corresponding to terms in the first and second be left out, the result is the same. For example, if the last terms $K_3$, $K_4$ are absent,

$$(A_1 + B_1 + C_1 + \ldots + K_1) : (A_3 + B_3 + C_3 + \ldots + I_3)$$
$$= (A_2 + B_2 + C_2 + \ldots + K_2) : (A_4 + B_4 + C_4 + \ldots + I_4),$$

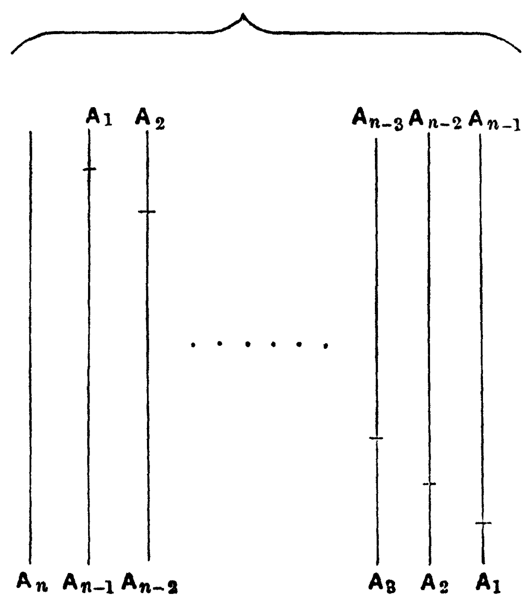
where $I$ immediately precedes $K$ in each series.

## Lemma to Proposition 2.

[*On Spirals*, Prop. 10.]

*If $A_1$, $A_2$, $A_3$, $\ldots A_n$ be $n$ lines forming an ascending arithmetical progression in which the common difference is equal to the least term $A_1$, then*

$$(n+1)A_n^2 + A_1(A_1 + A_2 + A_3 + \ldots + A_n)$$
$$= 3(A_1^2 + A_2^2 + A_3^2 + \ldots + A_n^2).$$




Let the lines $A_n$, $A_{n-1}$, $A_{n-2}$, $\ldots A_1$ be placed in a row from left to right. Produce $A_{n-1}$, $A_{n-2}$, $\ldots A_1$ until they are each equal to $A_n$, so that the parts produced are respectively equal to $A_1$, $A_2$, $\ldots A_{n-1}$.

Taking each line successively, we have

$$2A_n^2 = 2A_n^2,$$
$$(A_1 + A_{n-1})^2 = A_1^2 + A^2_{n-1} + 2A_1 \cdot A_{n-1},$$
$$(A_2 + A_{n-2})^2 = A_2^2 + A^2_{n-2} + 2A_2 \cdot A_{n-2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(A_{n-1} + A_1)^2 = A^2_{n-1} + A_1^2 + 2A_{n-1} \cdot A_1.$$

And, by addition,

$$(n+1)A_n^2 = 2(A_1^2 + A_2^2 + \dots + A_n^2)$$
$$+ 2A_1 . A_{n-1} + 2A_2 . A_{n-2} + \dots + 2A_{n-1} . A_1.$$

Therefore, in order to obtain the required result, we have to prove that

$$2(A_1 . A_{n-1} + A_2 . A_{n-2} + \dots + A_{n-1} . A_1) + A_1(A_1 + A_2 + A_3 + \dots + A_n)$$
$$= A_1^2 + A_2^2 + \dots + A_n^2 \quad \dots\dots\dots\dots (\alpha).$$

Now $\quad 2A_2 . A_{n-2} = A_1 . 4A_{n-2}$, because $A_2 = 2A_1$,

$2A_3 . A_{n-3} = A_1 . 6A_{n-3}$, because $A_3 = 3A_1$,

..........................

$2A_{n-1} . A_1 = A_1 . 2(n-1)A_1$.

It follows that

$$2(A_1 . A_{n-1} + A_2 . A_{n-2} + \dots + A_{n-1} . A_1) + A_1(A_1 + A_2 + \dots + A_n)$$
$$= A_1\{A_n + 3A_{n-1} + 5A_{n-2} + \dots + (2n-1)A_1\}.$$

And this last expression can be proved to be equal to

$$A_1^2 + A_2^2 + \dots + A_n^2.$$

For $\quad A_n^2 = A_1(n . A_n)$

$= A_1\{A_n + (n-1)A_n\}$

$= A_1\{A_n + 2(A_{n-1} + A_{n-2} + \dots + A_1)\}$,

because $(n-1)A_n = A_{n-1} + A_1$
$+ A_{n-2} + A_2$
$+ \dots\dots\dots\dots$
$+ A_1 + A_{n-1}$.

Similarly $A^2_{n-1} = A_1\{A_{n-1} + 2(A_{n-2} + A_{n-3} + \dots + A_1)\}$,

..........................

$A_2^2 = A_1(A_2 + 2A_1)$,

$A_1^2 = A_1 . A_1$;

whence, by addition,

$$A_1^2 + A_2^2 + A_3^2 + \dots + A_n^2$$
$$= A_1\{A_n + 3A_{n-1} + 5A_{n-2} + \dots + (2n-1)A_1\}.$$