## PART VI.

### FINAL EXPERIMENTS.

With the air condenser described in Part V., a number of spark photographs on Mr Swan's 4-inch square plates were taken with the plate revolving 64 times a second. From seven to nine circles were attempted on these plates with three or four sparks on each circle.

Tin plates which are lettered from *A* to *Z* were afterwards read with great care by Mr J. W. Capstick who writes: "The measurements will be found to be within a very few minutes of the correct reading. In one case I accidentally went over a spark twice, and though I was then at the end of six hours' almost continuous work at them, and the spark was an exceptionally indefinite one, the greatest divergence in the readings was only 3 or 4 minutes.

"The plates are very much better than any I had done previously, and the setting of the microscope was generally a simple matter. The sparks were in general so definite and regular that I did not think it necessary to make drawings of them."

[This had been done with some of the earlier plates.]

Mr Capstick remarks—as will be seen from the Tables—that there is some irregularity in the sparks, and that, unless it is desired to study this, greater accuracy of reading is hardly necessary.

The analysis of this long series of plates has been a work of time; we give below the results of a study of all the plates from *G* to *U*. In the earlier plates, marked *A* to *F*, the work was in some respects of a preliminary character; there was no plate marked *Q*. In the spin for plate *P* the coils were in multiple arc, and the coefficient of self-induction for this arrangement was not determined.

We give as an example the actual record for two of the circles on plate *U**.

This illustrates the method of dealing with the results.

SPARK RECORD ON PLATE *U*.

Coil *B* only used.

*Outer circle.*

| | Actual readings. | Differences. | Averages. |
|---|---|---|---|
| Spark (1) | 194° 0′ | | |
| | 186 49 | 14° 36′ | |
| | 179 24 | (14 45) | 14° 32′ |
| | (172 4) | 14 25 | |
| | 164 59 | (14 24) | |
| | 157 40 | | |

* The record for this plate happened to come first in one of the note-books in which results were recorded.

| | Actual readings. | Differences. | Averages. |
|---|---|---|---|
| Spark (2) | 155° 14′ | | |
| | (147 46) | 14° 21′ | |
| | 140 53 | 14 12 | 14° 22′ |
| | 133 34 | 14 33 | |
| | 126 20 | | |
| Spark (3) | 112° 40′ | | |
| | 106 4 | 14° 11′ | |
| or | 105 5 | (14 43) | |
| | 97 57 | 14 17 | 14° 18′ |
| | 90 48 | 14 9 | |
| | 83 48 | 14 12 | |
| | 76 36 | | |

General mean for this outer circle 14° 24′.

*Second circle.*

| | | | |
|---|---|---|---|
| Spark (1) | 136° 23′ | | |
| | 129 4 | 14° 12′ | |
| | 122 11 | 14 11 | |
| Spark (2) | 118 29 | | 14° 14′ |
| | 114 53 | 14 19 | |
| | 111 6 | | |
| | 107 52 | 14° 18′ | |
| | 104 11 | 14 10 | 14° 18′ |
| | 96 56 | 14 28 | |
| | 89 43 | | |
| Spark (3) | 175° 24′ | | |
| | 168 1 | 14° 18′ | |
| | (161 6) | 14 0 | 14° 13′ |
| | 154 2 | 14 21 | |
| | 146 25 | | |
| Spark (4) | blurred. | | |

Here there was some simple overlapping, giving no difficulty in sorting out. The general average for this circle is 14° 14′.

Table VIII.

Plate *U.* *Coil B.*

| Circle... | I. | II. | III. | IV. | V. | VI. | VII. | VIII. | IX. |
|---|---|---|---|---|---|---|---|---|---|
| Spark 1 | 14° 36′ | 14° 12′ | 14° 33′ | 14° 17′ | 14° 21′ | 14° 34′ | 14° 39′ | 14° 46′ | 14° 56′ |
| | 14 45 | 14 11 | 14 25 | 14 6 | 14 12 | 14 26 | 14 40 | 14 18 | 14 41 |
| | 14 25 | 14 19 | | 14 19 | 14 25 | 14 24 | 14 43 | 14 24 | 14 37 |
| | 14 24 | | | | | | | 14 56 | |
| 2 | 14 21 | 14 18 | 14 31 | 14 9 | 14 12 | 14 26 | 14 45 | 14 11 | 14 16 |
| | 14 12 | 14 10 | 14 18 | 14 5 | 14 10 | 14 14 | 14 23 | 14 14 | 14 12 |
| | 14 33 | 14 28 | 14 22 | 14 15 | 14 11 | 14 24 | 14 36 | 14 11 | 14 9 |
| | | | 14 49 | | | | 15 5 | | |
| 3 | 14 11 | 14 18 | 14 29 | 14 21 | 14 14 | 14 14 | 15 3 | 14 17 | 14 39 |
| | 14 43 | 14 0 | 14 19 | 14 20 | 14 20 | 14 5 | 14 29 | 14 20 | 14 10 |
| | 14 17 | 14 21 | | | 14 16 | 14 24 | 14 16 | 14 23 | 14 16 |
| | 14 9 | | | | | | 14 14 | | 14 21 |
| | 14 12 | | | | | | 14 7 | | |
| 4 | | | | 14 33 | 14 19 | 14 43 | | 14 23 | |
| | | | | 14 26 | 14 9 | 14 38 | | 14 20 | |
| | | | | 14 36 | 14 25 | 14 39 | | 14 34 | |
| Mean for circle | 14° 24′ | 14° 14′ | 14° 27′ | 14° 19′ | 14° 18′ | 14° 26′ | 14° 31′ | 14° 22′ | 14° 29′ |
| Mean of central swings for each circle | 14° 16′ | 14° 8′ | 14° 21′ | 14° 14′ | 14° 12′ | 14° 18′ | 14° 28′ | 14° 19′ | 14° 17′ |

General mean from plate 14° 23′.

Mean from central swings 14° 17′.

Thus in Table VIII. will be found the actual length in degrees and minutes of all the oscillations on the plate. The Roman numerals at the head of the columns indicate the circles on which the sparks are to be found; the record for each spark is shewn separately.

The mean length of oscillation from the 99 sparks here recorded is $14° 23'$; the range of the readings is rather over $1°$; the means for the various circles are given in the Table; they range over $17'$. It is clear however that the oscillations in any one spark are not of equal length. As a rule the first oscillation is a long one. This is followed by one or more of shorter period while, as the spark dies away, the oscillations again lengthen; the cause of this has been discussed in Part IV.

The lengthening of the latter oscillations is more plainly shewn on some of the other plates. If we omit the longer oscillations, and take only the more regular central swings on plate $U$, we get the following series of numbers, in which the $14°$ is omitted for brevity.

TABLE IX.

| Circle... | I. | II. | III. | IV. | V. | VI. | VII. | VIII. | IX. |
|---|---|---|---|---|---|---|---|---|---|
| | 25 | 11 | 25 | 6 | 12 | 26 | 40 | 18 | 41 |
| | 12 | 10 | 18 | | 10 | 14 | 23 | 24 | 12 |
| | 17 | 0 | 19 | 5 | 11 | 5 | 36 | 14 | 9 |
| | 9 | | | 20 | 16 | 38 | 29 | 11 | 10 |
| | | | | 26 | 9 | | 14 | 20 | 16 |
| | | | | | | | | 23 | |
| | | | | | | | | 20 | |
| Averages | $16'$ | $8'$ | $21'$ | $14'$ | $12'$ | $18'$ | $28'$ | $19'$ | $17'$ |

These lead to an average length of oscillation of $14° 17'$.

In taking the average in this manner we have given equal weight to each observation.

Now the record of this plate taken at the time of the observation is

PLATE $U$.

Air condenser in circuit with $B$ coil.

Machine not in circuit but arranged to charge it through a pair of needle points from a distance so that its capacity should not interfere.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| On the | outer | circle | were | taken | 4 | sparks, | speed | steady. |
| „ | second | „ | „ | „ | 4 | „ | „ | fair. |
| „ | third | „ | „ | „ | 3 | „ | „ | steady. |
| „ | fourth | „ | „ | „ | 4 | „ | „ | fair. |
| „ | fifth | „ | „ | „ | 4 | „ | „ | fair. |
| „ | sixth | „ | „ | „ | 4 | „ | „ | quite steady. |
| „ | seventh | „ | „ | „ | 4 | „ | „ | steady. |
| „ | eighth | „ | „ | „ | 4 | „ | „ | steady on average. |
| „ | ninth | „ | „ | „ | — | „ | „ | — — — |

(The number of sparks taken is not with perfect certainty correct, because there was sometimes a difficulty in hearing them.)

The remarks as to the speed were noted at the time according as the stroboscopic pattern had successfully been held still or not while the circle was being taken.

If attention is paid to these speed remarks it would seem that circles I., III., VI., VII. and VIII. should have most weight attached to them.

The averages for these circles are 16′, 21′, 18′, 28′, 19′, for the middle swings, and their mean is 14° 20′.

The complete averages for these steadiest circles are 24′, 27′, 26′, 31′, 22′, and the mean of these is 14° 26′.

It would thus appear that the best value for the wave length for this plate is **14° 20′**; while if all the sparks be included which lie on the circles retained, the number is increased by 6′; if all the circles are included, each of these numbers is reduced by 3′.

We may claim then to know the length of the oscillation on this plate to about 5′, i.e. to about ·6 %.

The frequency corresponding to 14° 20′ is $64 \times 360/14{\cdot}33$ or

**1608.**

## PLATE *S*.

Another series of wave lengths as recorded on plate *S*, in which coil *A* only was used, is given in Table X.

The notes relating to this plate are as follows.

On this plate the sparks photographed were taken from the air condenser through the *A* coil only. Machine charging via needle points.

TABLE X.

PLATE *S*. SUMMARY OF READINGS.

*Coil A*.

| Circle... | I. | II. | III. | IV. | V. | VI. | VII. | VIII. |
|---|---|---|---|---|---|---|---|---|
| | 14° 31′ | 14° 36′ | 14° 37′ | 14° 28′ | 14° 31′ | 14° 53′ | (blurred) | (all overlap) |
| | 14 11 | 14 20 | 14 29 | 14 3 | 14 2 | | 15° 6′ | |
| | 14 24 | 14 28 | 14 28 | 14 16 | 14 43 | 14 33 | 14 26 | |
| | 14 49 | 14 39 | 14 11 | | | 14 52 | 14 32 | |
| | | | | 14 15 | | | | |
| | 14 19 | 14 41 | 14 45 | 14 35 | 14 39 | | 14 56 | |
| | 13 57 | 14 21 | 14 43 | 13 58 | 14 4 | 14 19 | 14 22 | |
| | 14 11 | 14 22 | 14 17 | 14 9 | 14 11 | 14 22 | 14 14 | |
| | | 14 25 | | 14 28 | 14 51 | 14 48 | 14 2 | |
| | | 14 32 | ( ) | 14 35 | | | 14 40 | |
| | | 14 13 | | 14 15 | | | 14 33 | |
| | | 14 18 | 14 19 | 14 34 | | | 14 44 | |
| | | 14 26 | 14 47 | 14 47 | | | | |
| | | 14 38 | | 14 45 | | | | |
| | | 14 30 | | 14 19 | | | | |
| | | 14 26 | | 14 25 | | | | |
| | | 14 17 | | 14 50 | | | | |
| | | 14 42 | | | | | | |
| General Mean | 14° 22′ | 14° 28′ | 14° 28′ | 14° 26′ | 14° 26′ | 14° 38′ | 14° 27′ | |
| Mean of central waves | 14° 11′ | 14° 22′ | 14° 23′ | 14° 15′ | 14° 6′ | 14° 26′ | 14° 24′ | |

General mean 14° 28′.

Mean for centre swings 14° 19′.

Four sparks were taken on each circle.

| | | | | | |
|---|---|---|---|---|---|
| Circle I. | speed | moderately steady. | Circle V. | speed | fair. |
| II. | „ | fair. | VI. | „ | steady on average. |
| III. | „ | quite steady for 3 sparks. | VII. | „ | slightly backing. |
| IV. | „ | steady. | VIII. | „ | fair. |

To save space only the differences are quoted. All the differences read are included. Sometimes overlapping prevented any reading being attempted.

The general mean from Table X. is 14° 28′, while the central swings give 14° 19′. These means include all the circles. The range of the mean readings is about the same as for plate *U*, and the frequency calculated for the central swings works out to **1610** oscillations per second.

TABLE XI.

PLATE *R*. *Complete Coil A + B.*

| Circle... | I. | II. | III. | IV. | V. | VI., VII. |
|---|---|---|---|---|---|---|
| Spark 1 | 26° 52′<br>26 16<br>26 18<br>26 46 | 26° 53′<br>26 22 | 26° 34′<br>26 11 | 26° 43′<br>26 19 | 26° 47′<br>26 23 | |
| 2 | 26 44<br>26 4<br>26 51 | 26 55<br>26 18 | 26 51<br>26 15 | 26 52<br>26 20<br>27 7 | 27 1<br>26 23 | Overlapping one spark only |
| 3 | | 26 59<br>26 4 | 26 51<br>26 12<br>26 50 | 26 55 | 26 46<br>26 19<br>26 37 | |
| 4 | | | | 26 38<br>26 4 | | |
| General Mean | 26° 33′ | 26° 30′ | 26° 32′ | 26° 37′ | 26° 37′ | |
| Central Mean | 26° 12′ | 26° 15′ | 26° 13′ | 26° 14′ | 26° 21′ | |

General mean of plate 26° 34′.

Mean from central series 26° 15′.

As an example of a plate in which the whole coil was used the record for plate *R* is given in Table XI. It will be seen that the means for the separate circles differ by 7′ in the case in which all the sparks are considered, and by 9′ when only the central swings are dealt with; the difference between the two means is 19′.

If we take as the length of wave 26° 15′, the frequency is $64 \times 360/26{\cdot}25$ or 878 oscillations per second.

It is not necessary to give the results of the other plates in such full detail.

The following Table summarizes them sufficiently. In each case the central sparks only are included.

TABLE XII.

*Complete Coil A + B.*

| Plate | *K* | *L* | *O* | *R* | *T* |
|---|---|---|---|---|---|
| Number of sparks | 11 | 13 | 9 | 15 | 22 |
| Length of wave | 26° 14′ | 26° 9′ | 26° 16′ | 26° 15′ | 26° 5′ |

Mean length of wave 26° 11′.

*Coil A.*

| Plate | *G* | *H* | *M* | *N* | *S* |
|---|---|---|---|---|---|
| Number of sparks | 20 | 28 | 7 | 27 | 33 |
| Length of wave | 14° 14′ | 14° 32′ | 14° 25′ | 14° 18′ | 14° 19′ |

Mean length of wave 14° 20′.

*Coil B.*

| Plate | $J$ | $U$ |
|---|---|---|
| Number of sparks | 28 | 37 |
| Length of wave | 14° 26′ | 14° 20′ |

Mean length of wave 14° 22′.

From these we find as the mean length of the wave when the complete coil $A+B$ is used **26° 11′**.

With regard to the observations made with the coils $A$ and $B$ in circuit separately, it will be observed that plates $H$ and $J$ give higher results than the others. Now there is a note in the book that for these two series the outer plates of the condenser were earthed; they were taken therefore under different conditions to the others; if they be omitted we have as the mean wave length for plate $A$ **14° 18′**, and for $B$ **14° 20′**; if we include plates $H$ and $J$, the mean for $A$ is **14° 20′** and for $B$ **14° 22′**.

The corresponding frequencies are, excluding plates $H$ and $J$,

for coil $(A+B)$ **880** per second,

for coil $A$ **1611** per second,

for coil $B$ **1607** per second.

If we take the whole series of sparks for $A$ and $B$ we get respectively for $A$ 1607, and for $B$ 1603.

While the frequencies given by plates $H$ and $J$ are

for $A$ **1583**,

and for $B$ **1595**.

It is hardly necessary to work out the frequencies for each plate. For the complete coil $A+B$ the greatest variation from the mean is four parts in one thousand.

We may now determine from these spark records the value for "$v$."

We have the formula

$$v = 2\pi \,.\, \text{frequency} \times \sqrt{\frac{LS}{k}},$$

where $k$ is the constant, the values of which are given in Table V., occurring in the formula $LS\lambda^2 = k$. In the case in which the two coils were used there is no difficulty in deciding on the value of $k$. The formula for $\lambda$ is that given on p. 159 (D),

$$LS\lambda^2 = 1 - \frac{1}{3}\frac{S_1}{S}\left\{1 - \frac{1}{2(n-1)}\right\} + \frac{4}{45}\left(\frac{S_1}{S}\right)^2 + \ldots,$$

and hence $k = {\cdot}916$.

If only one coil is used two cases may arise; if the lower coil is completely insulated we have the case dealt with in Figure 4; the corresponding formula as far as terms in $S_1/S$ are concerned is (F) on p. 166, viz.:

$$LS\lambda^2 = 1 - \frac{1}{12}\left(1 + \frac{1}{n}\right)\frac{S_1}{S},$$

and the value of $k$ resulting from this is ·978. If on the other hand the lower coil is not insulated the correction necessary will be that indicated in (G), p. 167, and the resulting value of $k$ will be the same as that for the two coils, viz. ·916.

As far as we know the coil was usually insulated; at any rate it was not intentionally connected to earth except for the two plates $H$ and $J$.

But there is another complication in this case. We assume in this case that the value of $L$ is that for either half the coil; now this assumes that there is no current in the unused coil; but in consequence of the electrostatic induction there is a current in the unused coil. This current will be of the order $xS'/S$ if $x$ is the current from the main condenser, and its effect will therefore alter the coefficient of self-induction $L$ of the upper coil by an amount proportional to $MS'/S$ or about $M/120$. Now the value of $L_1$ is about 1·4, and of $M$ about ·91. Hence the value of $L_1$ in the experiments with the single coil is uncertain to one part in one hundred and seventy.

Omitting however this correction we get the following Table of values.

TABLE XIII.

| Coil used | $k$ | Frequency | $L$ | $S$ | $v$ | Observations |
|---|---|---|---|---|---|---|
| $A+B$ | ·916 | 880 | 4·636 | 58·53 | $3{\cdot}009 \times 10^{10}$ | Mean from seventy sparks |
| $A$ | ·978 | 1611 | 1·409 | 58·53 | $2{\cdot}939 \times 10^{10}$ | Unused coil assumed insulated |
| | ·916 | 1611 | 1·409 | 58·53 | $3{\cdot}037 \times 10^{10}$ | ,, ,, ,, uninsulated |
| | ·916 | 1583 | 1·409 | 58·53 | $2{\cdot}984 \times 10^{10}$ | Plate $H$, coil uninsulated |
| $B$ | ·978 | 1607 | 1·393 | 58·53 | $2{\cdot}922 \times 10^{10}$ | Unused coil assumed insulated |
| | ·916 | 1607 | 1·393 | 58·53 | $3{\cdot}020 \times 10^{10}$ | ,, ,, ,, uninsulated |
| | ·916 | 1595 | 1·393 | 58·53 | $2{\cdot}990 \times 10^{10}$ | Plate $J$, coil uninsulated |

In the fourth and seventh lines of this Table we give the velocity as obtained from plates $H$ and $J$. We know that in this case the effective coil and one plate of the condenser was earthed originally, and we have therefore used the value of $k$ calculated on the assumption that the free coil was earthed throughout. It will be seen that the resulting values of "$v$" and that obtained from the experiments with the full coil are in close agreement, being respectively $2{\cdot}98 \times 10^{10}$, $2{\cdot}99 \times 10^{10}$ and $3{\cdot}01 \times 10^{10}$.

If we take the other observations for coils $A$ and $B$, excluding plates $H$ and $J$, the results are not quite so satisfactory. The assumption that the free coil was insulated leads to the values $2{\cdot}94 \times 10^{10}$ and $2{\cdot}92 \times 10^{10}$, given in lines 2 and 5 of the Table; on the assumption that it was earthed we find from the same series of experiments the values $3{\cdot}04 \times 10^{10}$ and $3{\cdot}02 \times 10^{10}$ respectively, given in lines 3 and 6. The truth would appear to lie between the two.

If we take the experiments with the complete coil $A + B$ in series, we can determine the corrections with greater accuracy, and we find as the result

$$\mathbf{v} = \mathbf{3{\cdot}009} \times \mathbf{10^{10}} \text{ centimetres per second,}$$

while since the corrections can be calculated with more exactness in this case, we attach far greater importance to the result.

We do not however look upon the paper as one describing a very exact method of determining "$v$," but rather as a study in the oscillatory discharge of a condenser which incidentally leads to a determination of "$v$" by a novel method.

# VIII. *The Geometry of Kepler and Newton.* By Dr C. TAYLOR, Master of St John's College.

[*Received* 25 August, 1899.]

THIS paper consists of two parts (A) and (B), treating respectively of some things in the geometry of Kepler and some in the geometry of Newton, the finisher, in pure mathematics as in physics, of so much of his brilliant predecessor's work.

In Fontenelle's *Panegyrick* of Newton, published in French and English under the title, *The Life of Sir Isaac Newton with an Account of his Writings* (London, 1728), the third paragraph begins thus, "In studying Mathematicks, he employ'd his Thoughts very little upon *Euclid*, as judging him too plain and easy to take up any part of his time; he understood him almost before he had read him, and by only casting his eye upon the Subject of a Proposition, was able to give the Demonstration. He launch'd at once into such books as the Geometry of *Des Cartes* and the Opticks of *Kepler*. So that we may justly apply to him what Lucan has said of the Nile, whose Springs were unknown to the Antients, *That it was not granted to Mankind to see the* Nile *in a small Stream.*"

## (A)

## KEPLER.

Kepler's new and modern doctrine of the Cone and its sections, which historians of mathematics have ascribed to a later generation, was propounded in cap. IV. 4 of his *Ad Vitellionem Paralipomena, quibus Astronomiæ Pars Optica traditur,* a work published originally in 1604, a century before Newton's *Opticks* (1704), and edited with notes forty years ago by Dr Ch. Frisch in vol. II. of his *Joannis Kepleri Astronomi Opera Omnia* in eight volumes. The passage containing the new doctrine is given below line for line, with the addition of numbers for reference, from the edition of 1604 :

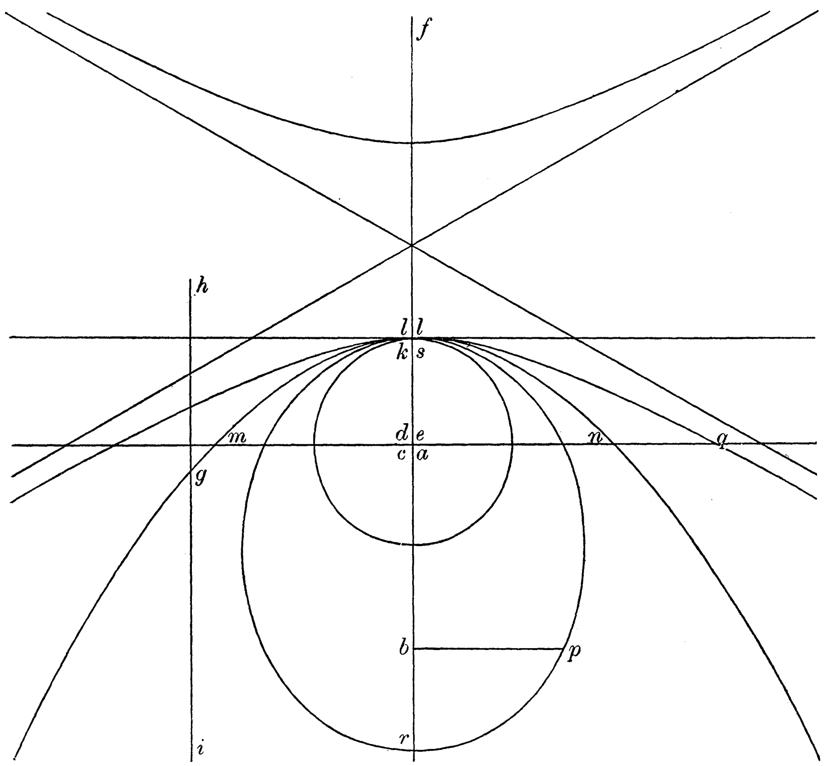
PAGE 92.

### 4. *De Coni ſectionibus.*

Coni varii ſunt, rectanguli, acutanguli, obtuſanguli: item Coni recti ſeu regulares, & Scaleni ſeu irregulares aut compreſſi: de quibus vide Apollonium & Eutocium in commentariis. Omnium promiſcuè ſectiones in quinq; cadunt ſpecies. Etenim linea in ſuperficie coni per ſectionẽ conſtituta aut eſt recta, aut circulus, aut Parabole aut Hyperbole aut Ellipſis. Inter has lineas hic eſt ordo cauſa proprietatis ſuæ: & analogicè magis quàm Geometricè loquendo: quod à linea recta per hyperbolas infinitas in Parabolen, inde per Ellipſes infinitas in circulum eſt tranſitus. Etenim omnium Hyperbolarum obtuſiſſima eſt linea recta, acutiſſima Parabole: ſic omnium Ellipſium acutiſſima eſt parabole, obtuſiſſima Circulus. Parabole igitur habet ex altera parte duas naturâ infinitas, Hyperbolen & Rectam, ex altera duas finitas, & in ſe redeuntes, Ellipſin & circulum. Ipſa PAGE 93. loco medio media natura ſe habet. Infinita enim & ipſa eſt, ſed finitionem ex altera parte affectat, quo magis enim producitur, hoc magis fit ſibiipſi parallelos, & brachia, vt ita dicam, non vt Hyperbole, expandit, ſed contrahit ab infiniti complexu, ſemper plus quidem complectens, at ſemper minus appetens: cum Hyperbole, quò plus actu inter brachia complectitur, hoc plus etiam appetat. Sunt igitur oppoſiti termini, circulus & recta, illic pura eſt curuitas, hic pura rectitudo. Hyperbole, Parabole, Ellipſis, interiectæ, & recto & curuo participant; parabole ex æquo, Hyperbole plus de rectitudine, Ellipſis plus de curuitate. Propterea Hyperbole quo longiùs producitur, hoc magis rectæ ſeu Aſymptoto ſuæ fit ſimilis. Ellipſis quò longiùs vltra medium continuatur, hoc magis circularitatem affectat, tandemq́ue coit iterum ſecum ipſa: Parabole loco medio, ſemper curuior eſt Hyperbola, ſi æqualibus interſtitiis producantur, ſemperq́ue rectior Ellipſi. Cumq́ue vt circulus & recta extrema claudunt, ſic Parabole teneat medium: ita etiam vt rectæ omnes ſimiles, itemq́ue & circuli omnes, ſic ſunt & parabolæ omnes ſimiles; ſolaq́ue quantitate differunt.

Sunt autem apud has lineas aliqua puncta præcipuæ conſiderationis, quæ definitionem certam habent, nomen nullum, niſi pro nomine definitionem aut proprietatem aliquam vſurpes. Ab iis enim punctis rectæ eductæ ad contingentes ſectionem, punctaq́; contactuum, conſtituunt æquales angulos iis, qui fiunt; ſi puncta oppoſita cum iisdem punctis contactuum connectantur. Nos lucis causâ, & oculis in Mechanicam intentis ea puncta Focos appellabimus. Centra dixiſſemus, quia ſunt in axibus ſectionum, niſi in Hyperbola & Ellipſi conici authores aliud punctum centri nomine appellarent. Focus igitur in circulo vnus eſt A. isq́ue idem qui & centrum: in Ellipſi foci duo ſunt BC. æqualiter à centro figuræ remoti & plus in acutiore. In Parabole vnus D eſt intra ſectionem, alter vel extra vel intra ſectionem in axe fingendus eſt infinito interuallo à priore remotus, adeò vt educta HG vel IG ex illo cæco foco in quodcunque punctum ſectionis G. ſit axi DK parallelos. In Hyperbola focus externus

PAGE 94.



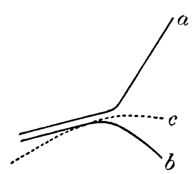

F interno E tantò eſt propior, quantò eſt Hyperbole obtuſior. Et qui externus eſt alteri ſectionum oppoſitarum, is alteri eſt internus & contra.

Sequitur ergò per analogiam, vt in recta linea vterque focus (ita loquimur de recta, ſine vſu, tantum ad analogiam complendam) coincidat in ipſam rectam: ſitque vnus vt in circulo. In circulo igitur focus in ipſo centro eſt, longiſſimè recedens à circumferentia proxima, in Ellipſi iam minus recedit, & in parabole multò minus, tandem in recta focus minimum ab ipſa recedit, hoc eſt, in ipſam incidit. Sic itaque in terminis, Circulo & recta, coëunt foci, illic longiſſimè diſtat, hic planè incidit focus in lineam. In media Parabole infinito interuallo diſtant, in Ellipſi & Hyperbole lateralib. bini actu foci, ſpatio dimenſo diſtant; in Ellipſi alter etiam intra eſt, in Hyperbole alter extra. Vndique PAGE 95. ſunt rationes oppoſitæ.

Linea MN quæ focum in axe metatur, perpendiculariter in axem inſiſtens, dicatur nobis chorda, & quæ altitudinem oſtendit foci à proxima parte ſectionis à vertice, pars nempe axis BR.

vel DK. vel E. S. dicatur Sagitta vel axis. Igitur in circulo ſagitta æquat ſemichordã, in Ellipſi maior eſt ſemichorda BF. q̃ ſagitta BR. maior etiam ſagitta BR. quàm dimidium BP ſemichordæ ſeu chordæ quarta pars. In Parabole, quod Vitellio demonſtrauit, ſagitta DK præcisè æquat quartam chordæ MN. hoc eſt D N eſt dupla ad DK. In Hyperbole EQ plus eſt, quàm dupla ipſius ES. ſc. minor eſt ſagitta ES. q̃ quarta chordæ EQ. & ſemper minor, atque minor per omnes proportiones, donec euaneſcat in recta, vbi foco in lineam ipſam incumbente, altitudo foci ſeu ſagitta euaneſcit, & ſimul chorda infinita efficit, coincidens ſc. cum arcu ſuo, abuſiuè ſic dicto, cùm recta linea ſit. Oportet enim nobis ſeruire voces Geometricas analogiæ: plurimùm namque amo analogias, fideliſſimos meos magiſtros, omnium naturæ arcanorum conſcios: in Geometria præcipuè ſuſpiciendos, dum infinitos caſus interiectos intra ſua extrema, mediumque, quantumuis abſurdis locutionibus concludunt, totamque rei alicuius eſſentiam luculenter ponunt ob oculos.

Quin etiam in deſcriptione ſectionum analogia plurimùm me iuuit. Etenim ex 51. & 52. tertii Apollonii deſcriptio Hyperboles & Ellipſeos efficitur facilima; poteſtque vel filo perfici. Poſitis enim focis, & inter eos vertice C. figantur acus in focis A. B. annectatur ad acum A filum longitudine AC. ad B. filum longitudine BC. Prolongetur vtrumque filum æqualibus additionibus, vt ſi duplex filum digitis comprehendas, iisque à C diſcedentibus, bina fila paulatim dimittas, alteraque manu ſignes iter anguli, quem vtrumque filum facit apud digitos, ea deſignatio erit hyperbole. Facilius Ellipſis deſcribitur. Foci ſint AB. vertex C. Fige acus firmas in A.B. vtramque filo amplectere, ſimplici amplexu, vt inter AB filum non interſit. Fili longitudo ſit AC duplicata, & capita fili nodo ſint connexa. Infere iam Graphium D in eundem fili complexum cum AB. & tenſo filo, quantum id patitur, circa AB circumduc lineam, hæc Ellipſis erit. Cùm hæc tam facilis eſſet deſcriptio, non indigens operoſis illis circinis, quibus aliqui cudendis admirationem hominum venantur; diu dolui, non poſſe ſic etiam Parabolen deſcribi. Tandem analogia mõſtrauit, (& Geometrica comprobat) non multò operoſiùs & hanc deſignare. Proponatur A focus, C vertex, vt AC ſit axis; is continuetur in partes A. in infinitum vſq;, aut quousq; Parabolen placuerit deſcribere. Placeat vſq; in E. Acus ergò in A figatur, ab ea ſit nexum filũ longitudine AC. CE. Teneas manu altera caput alterũ fili E. altera graphium, cũ filo extende vſq; in C. Sit etiam ad CE. erecta perpendiculariter EF.

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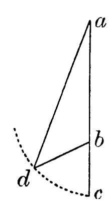


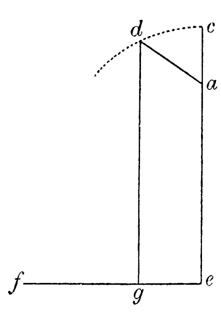

igitur graphio C & manu altera E diſcede æqualibus interuallis à linea AE. ſic vt manus altera & fili caput ſemper in EF maneat, filumq́ue DG ſemper ipſi AE parallelon; via CD. quam Graphio ſignaueris, erit Parabole.

Dixi hæc de ſectionibus conicis tanto libentiùs, quòd non tantùm hic dimenſio refractionum id requirebat, ſed etiam infra in Anatome oculi vſus earum apparebit. Tum etiam inter problemata obſeruatoria mentio earum erit facienda duobus locis. Denique ad præſtantiſſima optica machinamenta, ad penſilem in aëre ſtatuendam imaginem, ad imagines proportionaliter augendas, ad ignes incendendos, ad infinitè comburendum, conſideratio earum planè eſt neceſſaria.

*Machinamẽta Optica Portæ.*

The headlines of the edition quoted are *Ioannis Kepleri* and *Paralipom. in Vitellionem* up to page 221, and afterwards *Ioannis Kepleri* and *Astronomiæ Pars Optica.*

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Kepler begins by saying that rays from the centre of a sphere do not become parallel after reflexion from its inner surface, but converge to the centre. Some other surface then had to be sought which would reflect all rays from some point into parallels. Vitellio in lib. IX. 39—44, in part supplying what was lacking in Apollonius, had shewn that the paraboloid of revolution was of the required form. But the subject of the Conic Sections presented difficulties because it had not been much studied. Kepler therefore—pardon a geometer—proposed to discourse somewhat "mechanically, analogically and popularly" about them.

Vitellio or Vitello (Witelo) had proved that at any point of a parabola the tangent makes equal angles with a parallel to the axis and the line from the point to a certain fixed point on the axis. Rays of the sun impinging equidistantly from the axis upon the concavity of a reflecting paraboloid of revolution would therefore all be reflected through a fixed point on the axis, and fire might so be kindled thereat.

Of cones right or scalene there are five species of sections (line 24), the right line or *line-pair*, the circle, parabola, hyperbola and ellipse. From the line-pair we pass through an infinity of hyperbolas to the parabola, and thence through an infinity of ellipses to the circle. Of all hyperbolas the most obtuse is the line-pair, the most acute the parabola. Of all ellipses the most acute is the parabola, the most obtuse the circle.

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The parabola is of the nature partly of the infinite sections and partly of the finite, to which it is intermediate. As it is produced it does not spread out its arms in direction like the hyperbola, but contracts them and brings them nearer to parallelism, "semper plus quidem complectens at semper minus appetens" (line 5). The hyperbola being produced tends more and more to the form of its "Asymptote" (line 12). Parabolas are all similar and differ only in "quantity" (line 19).

He then goes on to speak of certain remarkable points related to the sections which had NO NAME (line 21). The lines from them to any point of the section make equal angles with the tangent. He will call them FOCI (line 27). He would have called them centres if that term had not been already appropriated. The circle has one focus, at the centre: the ellipse has two, equidistant from the centre, and more remote as the curve is more acute. In the parabola one is within the curve, while the other may be regarded as either without or within it, so that a line *hg* or *ig* drawn from that "cæcus focus" to any point of the curve is parallel to the axis (line 35).

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In the hyperbola the focus external to either branch is the nearer to its internal focus as the hyperbola is more obtuse. In the straight line (or line-pair), to speak in an unusual way merely to complete the analogy, the foci fall upon the line itself. Thus in the extreme limiting cases of the circle and the line-pair, the foci come together at a point, which in the one is as far as possible from the nearest point of the circumference and in the other is on the line itself. In the intermediate case of the parabola the foci are infinitely distant from one another (line 12): in the ellipse and the hyperbola on either side of it they are a finite distance apart.

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The line *mn* through the focus, i.e. the latus rectum, is called the *chord*, and *br* or *dk* or *es* the *sagitta* (line 6). In the next line *BF* is a misprint for *BP*. The lengths of the sagitta and the chord are compared in the five sections, and it is said that in the line-pair the one vanishes and the other becomes infinite (line 15), whereas, if $e$ be the eccentricity, they are in the finite ratio $1/2\,(1+e)$, and vanish together. Kepler commends the principle of analogy in glowing terms, saying that he dearly loves analogies, his most trusty teachers and conversant with all the secrets of nature (line 19). Analogy leads us to comprise in one definition extreme limiting forms, from the one of which we pass to the other by continuous variation through an infinity of intermediate cases.

In the next paragraph Kepler shews how to describe an arc of a hyperbola by means of threads fixed at the foci, the difference of the focal distances of a point on the curve being constant. An ellipse is described more easily (line 33), with one thread.

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In line 1 "AC duplicata" is inaccurate, the length of the thread being $ac+cb$. He is shewing how to describe an ellipse by means of a thread fixed at the foci $a$ and $b$, the point $c$ being a vertex. Having given his construction for this curve without the troublesome compasses (line 6), he goes on to the parabola. To his grief he was long unable to describe this analogously. At length he thought of the construction in the text, in which *adg* represents a string of constant length $ec+ca$ fixed at the focus $a$.

The horizontal line is a fixed ordinate, $c$ is the vertex and $d$ any point of the locus. His construction assumes a case of the theorem that the sum or difference of the distances of a point on the parabola from the focus and a fixed perpendicular to the axis is constant.

In conclusion he refers to later passages for applications of his theory of the conic sections. See cap. V. *De modo visionis,* and cap. XI. prob. 22—23 (p. 375 sq.).

## The Convergence of Parallels.

Vitellio, as we have seen, had proved that rays of the sun impinging equidistantly from (i.e. parallel to) the axis upon a concave reflector of the form of a paraboloid of revolution would all be reflected to a certain point on the axis, whereat consequently "ignem est possibile accendi." Hence in different languages the name "burning point" for what Kepler called *Focus,* in a parabola or other conic.

It would appear that the idea of the meeting of parallels at infinity came from the observed fact that solar rays received upon a reflector may practically be regarded as parallel. Moreover it was obvious that the distance, estimated on an infinitely remote transversal, between "equidistant" lines would subtend a vanishing angle at an assumed point of observation. Kepler does not say that his doctrine of parallels is altogether new and strange, when he writes at the end of page 93, "adeo ut...", *so that* lines from the point $h$ (or $i$) are parallel,—as if that would be allowed to follow from its being infinitely distant. But it was perhaps a new and original suggestion that $h$ and $i$ at infinity were the same point.

Kepler states expressly that he gave the name Foci to certain points related to the conic sections which had previously "no name." With their new name he associated his new views about the points themselves, and his doctrines of Continuity (under the name Analogy) and Parallelism, which would soon have become known, and would after a time have been taken up by competent mathematicians.

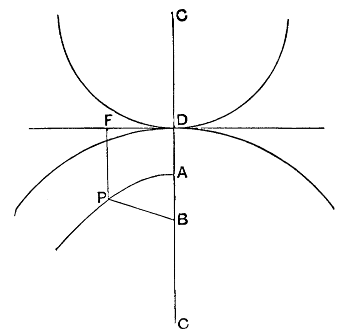
An abstract of the passage now quoted at length from Kepler's *Paralipomena ad Vitellionem* was given by the writer in *The Ancient and Modern Geometry of Conics**, published early in 1881, and previously in a note read in 1880 to the Cambridge Philosophical Society (*Proceedings,* vol. IV. 14—17, 1883), both of which have been referred to by Professor Gino Loria in his writings on the history of geometry.

## Henry Briggs.

Frisch (II. 405 sq.) quotes a letter of Henry Briggs to Kepler dated, Merton College, Oxford, "10 Cal. Martiis 1625," which suggests improvements in the *Paralipomena ad Vitellionem.* In this letter Briggs gives the following construction. Draw a line $CBADC$, and suppose an ellipse, a parabola and a hyperbola to have $B$ for focus and $A$ for their nearer vertex. Let $CC$ be the other foci of the ellipse and the hyperbola. Make $AD$ equal to $AB$, and with centres $CC$ and radius in each case equal to $CD$ describe circles. Then any point of the ellipse is equidistant from $B$ and one

* *The Ancient and Modern Geometry of Conics* is hereinafter referred to as *AMGC.*

circle, and any point of the hyperbola from $B$ and the other circle. When $C$ is at infinity on either side of $D$ the circle about it becomes rectilinear. Hence any point $P$ of the parabola is equidistant from $B$ and the perpendicular $DF$ to $DA$. This is expressed by Briggs as follows:




"Si $A$ sit vertex sectionis, et $B$, $C$ foci, et $AB$, $AD$ aequales, et centro $C$, radio $CD$ describatur peripheria: quodlibet punctum sectionis eandem servabit distantiam a foco $B$ et dicta peripheria. Eruntque...in Parabola (*cui focus alter deest, vel distat infinite, et idcirco recta $DF$ vicem obtinet peripheriae*) $PB$, $FP$ aequales."

The writer comprehended and accepted Kepler's way of looking at parallels as lines to or from a point at infinity in one direction or its opposite.

## DESARGUES.

The famous geometer Desargues worked on the lines of Kepler, and is now commonly credited with the authorship of some of the ideas of his predecessor.

Poncelet in the first edition of his *Traité des Propriétés Projectives des Figures* (1822) writes with reference to a letter of Descartes, "On voit aussi, dans cette lettre, que Desargues avait coutume de considérer les systèmes de droites parallèles comme concourant à l'infini, et qu'il leur appliquait le même raisonnement" (p. xxxix.). Chasles on the Porisms of Euclid refers to this remark of Poncelet. In his *Aperçu Historique* (p. 56, 1875) he writes that Kepler "introduisit, le premier, l'usage de l'*infini* dans la Géométrie," but really with reference only "aux méthodes infinitésimales." The saying that Kepler introduced the use of the infinite into geometry has been repeated by other writers unacquainted with his doctrine of the infinitely great.

Dr Moritz Cantor in his *Vorlesungen über Geschichte der Mathematik* writes under the head of Girard Desargues (1593—1662), "Wir müssen einige wesentliche Dinge hervorheben und darunter zunächst die Anwendung des Unendlichen in der Geometrie...Auch Kepler hat 1615, Cavalieri 1635 in Druckwerken, deren Besprechung uns obliegen wird, wenn wir von den Anfängen der Infinitesimalrechnung reden, den gleichen Gedanken zu nie geahnten Folgerungen ausgebeutet, aber bei Desargues waren es *ganz andere Unendlichkeitsbetrachtungen* als bei diesen Vorgängern" (II. 619, 1892). He goes on to say that Desargues regarded parallels as meeting at infinity, and thus in effect that Kepler did not so regard them. Cantor (p. 620 n.), referring to Poudra's

*Œuvres de Desargues* I. 103, states confidently that Desargues could not have held that "es gebe nur einen Unendlichkeitspunkt einer Gerade." "Auch in I. 105...darf man *jenen modernen Sinn* nicht hineinlesen." But the oneness of opposite infinities followed simply and logically from a first principle of Desargues, that every two straight lines, including parallels, have or are to be regarded as having one common point and one only. A writer of his insight must have come to this conclusion, even if the paradox had not been held by Kepler, Briggs, and we know not how many others, before Desargues wrote.

In Poudra's *Œuvres de Desargues,* I. 210, under the head *Traité des Coniques,* we read, "*Nombrils, point brulans, foyers.*—C'est à dire que les deux points comme $Q$ et $P$ sont les points nommés nombrils, brulans, ou foyers de la figure, au suiet desquel il y a beaucoup à dire." Desargues must have learned directly or indirectly from the work in which Kepler propounded his new theory of these points, first called by him the Foci (*foyers*), including the modern doctrine of real points at infinity.

## (B)

## NEWTON.

In the fifth section of the first book of the *Principia,* entitled *Inventio orbium ubi umbilicus neuter datur*, the determination of conic orbits from data not including a focus, Newton proves the property of the *Locus ad quatuor lineas* of which no geometrical demonstration was extant, shews how to describe conics by rotating angles and otherwise, and solves the six cases of the problem to determine a conic of which $n$ points and $5-n$ tangents are given. Two more problems, each with its Lemma prefixed, complete the section, which ends with the words, "Hactenus de orbibus inveniendis. Superest ut motus corporum in orbibus inventis determinemus."

The following pages contain a summary of the greater part of the section, with suggestions for the simplification of some of its contents and a few additional constructions and propositions. The Lemmas and Propositions of the *Principia* are quoted by their Roman numerals.

### 1.

### THE CONIC THROUGH FIVE POINTS.

PROP. A. *Given five points of a conic to find a sixth.*

Let $A$, $B$, $C$, $D$, $P$ be given points of a conic. Through $P$ draw $PTSO$ parallel to $BA$ across $BD$, $AC$, $CD$. It is required to find the point $K$ in which it meets the conic again.