where γ is a positive small quantity of the first order. When $n=0$,

$$l+m-\sigma-(l+m-1)(2-\sigma)=\epsilon(2-l-m),$$

so that as $l+m \geqslant 2$, we have

$$\frac{l+m-\sigma}{l+m-1}=2-\sigma-\gamma',$$

where γ' is a positive small quantity of the first order, unless $l+m=2$, and then $\gamma'=0$. Hence

$$\begin{aligned}\frac{P'}{P} &= \Pi\frac{1}{1-\gamma}\Pi\frac{1}{2-\sigma-\gamma'}\\ &> \Pi\frac{1}{2-\sigma}\\ &> \frac{1}{(2-\sigma)^{3l+2m+n}},\end{aligned}$$

the difference between the two sides being a small quantity of the first order. Also

$$\frac{Q\beta}{P'}$$

is a small quantity of the second order, that is, a quantity of an order less than the foregoing difference; consequently

$$\frac{P'}{P+Q\beta}>\frac{1}{(2-\sigma)^{3l+2m+n}}.$$

The changes depreciated the numerator of T into that of T': hence

$$\begin{aligned}\frac{T'}{T} &< \frac{P+Q\beta}{P'}\\ &< (2-\sigma)^{3l+2m+n}\\ &< (2-\sigma)^{3l+3m+3n}.\end{aligned}$$

This result holds for every term in h_{lmn}; hence

$$\left|\frac{h'_{lmn}}{h_{lmn}}\right|<(2-\sigma)^{3l+3m+3n}.$$

Similarly,

$$\left|\frac{k'_{lmn}}{k_{lmn}}\right|<(2-\sigma)^{3l+3m+3n}.$$

Let the region of convergence of the power-series

$$\Sigma\Sigma\Sigma h_{lmn}t^l\theta^m\phi^n, \quad \Sigma\Sigma\Sigma k_{lmn}t^l\theta^m\phi^n$$

be defined by the ranges

$$t \leqslant r, \quad |\theta| \leqslant r_1, \quad |\phi| \leqslant r_2;$$

and let M_1, M_2 be the maximum values of the moduli of the series respectively within this region; then

$$h_{lmn} < \frac{M_1}{r^l r_1^m r_2^n},$$

$$k_{lmn} < \frac{M_2}{r^l r_1^m r_2^n};$$

consequently

$$h'_{lmn} < \frac{M_1}{\left\{\frac{r}{(2-\sigma)^3}\right\}^l \left\{\frac{r_1}{(2-\sigma)^3}\right\}^m \left\{\frac{r_2}{(2-\sigma)^3}\right\}^n},$$

$$k'_{lmn} < \frac{M_2}{\left\{\frac{r}{(2-\sigma)^3}\right\}^l \left\{\frac{r_1}{(2-\sigma)^3}\right\}^m \left\{\frac{r_2}{(2-\sigma)^3}\right\}^n}.$$

Hence the series

$$\Sigma\Sigma\Sigma h'_{lmn} t^l \theta^m \phi^n, \quad \Sigma\Sigma\Sigma k'_{lmn} t^l \theta^m \phi^n,$$

converge absolutely for values of t such that $|t| < r$.

The existence of integrals of

$$\left.\begin{aligned} t\frac{dx}{dt} &= x + at + \Sigma\Sigma\Sigma a_{ijp} x^i y^j t^p \\ t\frac{dy}{dt} &= y + bt + \Sigma\Sigma\Sigma b_{ijp} x^i y^j t^p \end{aligned}\right\}$$

can be deduced from the preceding result, by choosing

$$|a| = A, \quad |b| = B, \quad |a_{ijp}| = A_{ijp}, \quad |b_{ijp}| = B_{ijp},$$

as the quantities A, B, A_{ijp}, B_{ijp} for those dominant equations. The expression for the integrals is

$$\left.\begin{aligned} x &= \Sigma\Sigma\Sigma H_{lmn} t^l \theta^m \phi^n \\ y &= \Sigma\Sigma\Sigma K_{lmn} t^l \theta^m \phi^n \end{aligned}\right\},$$

where H_{lmn} is derived from h'_{lmn}, and K_{lmn} from k'_{lmn}, by changing A into $-a$, B into $-b$, A_{ijp} into a_{ijp}, and B_{ijp} into b_{ijp}. The effect of these changes is to give

$$|H_{lmn}| < h'_{lmn},$$

$$|K_{lmn}| < k'_{lmn};$$

and therefore the series for x and y converge absolutely.

The actual values are

$$\left.\begin{aligned} x &= at\log t + C_1 t + \Sigma\Sigma\Sigma\, H_{lmn} t^l \theta^m \phi^n \\ y &= bt\log t + C_2 t + \Sigma\Sigma\Sigma\, K_{lmn} t^l \theta^m \phi^n \end{aligned}\right\},$$

where $\theta = -t\log t$, $\phi = \frac{1}{2}t(\log t)^2$, the summation is for values of l, m, n such that $l+m+n \geqslant 2$, and the coefficients C_1, C_2 are arbitrary constants.

But the formal expression is more general than the actual value. The equations determining the coefficients are

$$\left.\begin{aligned}(l+m+n-1)\,H_{lmn}-(m+1)\,H_{l-1,\,m+1,\,n}-(n+1)\,H_{l,\,m-1,\,n+1}&=E_{lmn}\\(l+m+n-1)\,K_{lmn}-(m+1)\,K_{l-1,\,m+1,\,n}-(n+1)\,K_{l,\,m-1,\,n+1}&=F_{lmn}\end{aligned}\right\},$$

with

$$H_{100}=C_1,\quad H_{010}=-a,\quad H_{001}=0,$$

$$K_{100}=C_2,\quad K_{010}=-b,\quad K_{001}=0.$$

It is clear that, when $l+m+n=2$,

$$E_{lmn}=0,\ \ F_{lmn}=0,\ \text{ if }\ n=1,\ 2\,;$$

hence H_{lmn}, K_{lmn} both vanish for $l+m+n=2$ if $n=1,\ 2$.

Thus for $l+m+n=3$,

$$E_{lmn}=0,\ \ F_{lmn}=0,\ \text{ if }\ n=1,\ 2,\ 3\,;$$

hence also H_{lmn}, K_{lmn} both vanish for $l+m+n=3$ if $n=1,\ 2,\ 3$. And so on: all the coefficients H_{lmn}, K_{lmn} vanish if

$$n>0\,;$$

that is, the quantity ϕ does not actually occur in the expressions for x and y which accordingly are regular functions of t and $t\log t$.

The theorem is therefore established.

Note 1. Any term in x and y is of the form

$$Kt^m\,(t\log t)^n,$$

that is, $Kt^{m+n}\,(\log t)^n$; and therefore the index of $\log t$ is never greater than the index of t.

If, however, the equations were

$$\left.\begin{aligned}t\,\frac{dx}{dt}&=x+at+ct\log t+\Sigma\Sigma\Sigma\Sigma a_{ijpq}\,x^iy^jt^p\,(t\log t)^q\\t\,\frac{dy}{dt}&=y+bt+c't\log t+\Sigma\Sigma\Sigma\Sigma b_{ijpq}\,x^iy^jt^p\,(t\log t)^q\end{aligned}\right\},$$

where $i+j+p+q\geqslant 2$ for the summations, then the values of x and y satisfying the equations are

$$\left.\begin{aligned}x&=-\tfrac{1}{2}ct\,(\log t)^2+at\log t+C_1t+\Sigma\Sigma\Sigma H_{lmn}\,t^l\theta^m\phi^n\\y&=-\tfrac{1}{2}c't\,(\log t)^2+bt\log t+C_2t+\Sigma\Sigma\Sigma K_{lmn}\,t^l\theta^m\phi^n\end{aligned}\right\},$$

where t, θ, ϕ have the same values as above, and the summations are for values of

l, m, n such that $l+m+n \geqslant 2$: and the coefficients H_{lmn}, K_{lmn} are determinable as before. Any term in x is

$$Ht^{l+m+n} (\log t)^{m+2n},$$

that is, the index of $\log t$ is not greater than twice the index of t.

Note 2. If a vanishes but not b, the solutions are still non-regular functions of t; likewise if b vanishes but not a. In these cases, it is known that no regular integrals vanishing with t are possessed by the equation.

If $a=0$, $b=0$, then $H_{lm}=0$, $K_{lm}=0$, if $m \geqslant 1$: that is, $t \log t$ disappears from the expressions for x and y, which then become regular functions and are the double infinitude of regular integrals that vanish with t. In this case, the regular integrals are the only integrals vanishing with t that are possessed by the equation.

20. *Second sub-case: κ not zero.*

The theorem is:

The equations possess in general a double infinitude of non-regular integrals vanishing with t which are regular functions of t, $t \log t$, $\frac{1}{2} t (\log t)^2$; and it is known that there are no regular integrals which vanish with t. If however $a=0$, then the integrals can be arranged in two sets; one is a simple infinitude of non-regular integrals vanishing with t which are regular functions of t and $t \log t$; the other is the simple infinitude of regular integrals vanishing with t which the equation is known to possess. (It is necessary that the constant κ be different from zero: otherwise some of the coefficients in the second set are infinite unless b also is zero, in which form we revert to the first sub-case already considered.)

The method of establishment is similar to those which precede: it need therefore not be repeated after the many instances of it which already have been given.

The initial terms in the integrals of the equations as taken in § 15 are

$$t_1 = a\theta + At + \ldots,$$

$$t_2 = \kappa a \phi + (\kappa A + b)\theta + Bt + \ldots,$$

the unexpressed terms being of higher order in t, θ, ϕ: here A and B are arbitrary, $\theta = t \log t$, and $\phi = \frac{1}{2} t (\log t)^2$. Any term in the expansion of t_1 or t_2 which involves ϕ contains κ in its coefficient; the disappearance of the terms in ϕ from the integrals in the first sub-case is thus explained, for κ then is zero.

Concluding Note.

21. Some sub-cases still remain over from Case I (a), when the roots ξ_1 and ξ_2 of the critical quadratic do not satisfy the conditions that (§ 8) prevent some one (or more) of the quantities

$$(\lambda-1)\xi_1 + \mu\xi_2 + \nu, \quad \lambda\xi_1 + (\mu-1)\xi_2 + \nu,$$

from vanishing for integer values of λ, μ, ν such that $\lambda + \mu + \nu \geqslant 2$. The real parts of ξ_1, ξ_2 are supposed to be positive.

The instances that can occur are obviously for $\lambda = 0$ in the first set and $\mu = 0$ in the second set; both are included in the form

$$\xi = \mu\eta + \nu,$$

where ξ and η are the roots of the quadratic, and $\mu + \nu \geqslant 2$. The cases $\mu = 0$, $\mu = 1$, have already been discussed. For the remaining cases, we have the theorem: *The double infinitude of non-regular integrals vanishing with t are then regular functions of t, t^η, $t^{\mu\eta+\nu} \log t$, where μ and ν are integers.* It can be established in the same manner as the similar theorems in the preceding sections.

IV. *Ueber die Bedeutung der Constante b des van der Waals'schen Gesetzes.* Von PROF. BOLTZMANN und DR MACHE, in Wien.

[Received 1899 August 14.]

IN dem Buche von Professor Boltzmann "Vorlesungen über Gastheorie, II. Theil" wurde die *van der Waals'sche* Formel aus der Vorstellung abgeleitet, dass die Gasmoleküle Anziehungskräfte auf einander ausüben, deren Wirkungssphäre gross ist gegen den Abstand zweier Nachbarmoleküle. Der Fall, wo diese Annahme nicht mehr zutrifft, wurde in demselben Buche auf Seite 213 kurz behandelt. Es zeigt sich, dass dann Erscheinungen, wie sie bei der *Dissociation* zweiatomiger Gase vorkommen, nicht eintreten können, falls die Anziehungskraft gleichmässig nach allen vom *Atomcentrum* ausgehenden Richtungen wirkt. Die an jener Stelle abgeleiteten Formeln können aber benützt werden, um die Zustandsgleichung zu entwickeln. Es wurde dort die Annahme gemacht, dass die daselbst mit χ bezeichnete Grösse *constant* ist. Lassen wir diese Annahme fallen, so tritt an Stelle der Formel 233 allgemein der Ausdruck

$$\frac{n_2}{n_1} = \frac{2\pi n_1}{V} \int_{\sigma}^{\sigma+\delta} r^2 dr\, e^{2hf(r)}.$$

Es wird also jetzt angenommen, dass die Trennungsarbeit von der Tiefe abhängig ist, bis zu welcher das *Centrum* eines zweiten Moleküls in den kritischen Raum des ersten eingedrungen ist. Dagegen soll zunächst der Fall dahin vereinfacht werden, dass die Anziehungskraft innerhalb dieses kritischen Raumes *constant* bleibt. Dann wird

$$f(r) = C(\sigma + \delta - r).$$

Schreibt man zur Abkürzung $2hC = c$ und führt die *Integration* durch, so hat man

$$n_2 = \frac{2\pi n_1^2}{Vc^3}\{e^{c\delta}[(c\sigma+1)^2+1] - [(c\overline{\sigma+\delta}+1)^2+1]\} = \frac{\kappa}{2} n_1^2.$$

Es gilt aber allgemein für ein Gasgemisch aus n_1 und n_2 Molekülen verschiedener Art die Beziehung

$$pV = \frac{m\overline{c^2}}{3}(n_1+n_2) = MRT(n_1+n_2).$$

Nennen wir a die Zahl der Moleküle bei vollkommener *Dissociation*, so ist

$$a = n_1 + 2n_2 = n_1 + \kappa n_1^2.$$

Hingegen ist die Zahl der freien Moleküle im betrachteten Zustand

$$n = n_1 + n_2 = \frac{a + n_1}{2}.$$

Durch Elimination von n_1 und Entwickeln der Wurzel findet man hieraus den Näherungswert $n = a - \frac{a^2\kappa}{2}$ und folglich auch weiters

$$pV = aMRT - \frac{a^2MRT}{2}\kappa.$$

Ist aber m die Masse eines Moleküls, μ das Atomgewicht, v das *specifische* Volumen, endlich r die Gas*constante* des betrachteten Gases, so ist $M = \frac{m}{\mu}$, $\frac{am}{V} = \frac{1}{v}$, endlich $\frac{R}{\mu} = r$ und es wird auch

$$p = \frac{rT}{v} - \frac{arT}{2v}\kappa,$$

oder wenn man auf den Ausdruck für κ zurückgeht

$$p = \frac{rT}{v} - \frac{1}{v^2} \cdot \frac{2\pi rT}{c^3 m}\{e^{c\delta}[(c\sigma + 1)^2 + 1] - [(\overline{c\sigma + \delta} + 1)^2 + 1]\} = \frac{rT}{v} - \frac{A}{v^2}.$$

Hiebei ist aber in v noch der von den Deckungsphären der Moleküle ausgefüllte Raum $\rho = \frac{1}{m} \cdot \frac{4}{3}\pi\sigma^3$ abzuziehen. Wir erhalten also als Zustandsgleichung

$$p = \frac{rT}{v - \rho} - \frac{A}{(v - \rho)^2}.$$

Zur Discussion dieser Formel finde noch folgende Betrachtung Raum. Es ist, wie man sich leicht durch Rechnung überzeugt,

$$e^{c\delta}[(c\sigma + 1)^2 + 1] - [(\overline{c\sigma + \delta} + 1)^2 + 1] = c^3\sigma^2\delta \sum_{n=1}^{n=\infty}(c\delta)^{n-1}\left\{\frac{1}{n\,!} + \frac{2\,\frac{\delta}{\sigma}}{\overline{n+1}\,!} + \frac{2\left(\frac{\delta}{\sigma}\right)^2}{\overline{n+2}\,!}\right\}.$$

Ferner ist
$$A = \frac{1}{m} \cdot 2\pi\sigma^2\delta rT \sum_{n=1}^{n=\infty}(c\delta)^{n-1}\left\{\frac{1}{n\,!} + \frac{2\,\frac{\delta}{\sigma}}{\overline{n+1}\,!} + \frac{2\left(\frac{\delta}{\sigma}\right)^2}{\overline{n+2}\,!}\right\}.$$

Es gilt weiters die Beziehung $c = 2hC = \frac{C}{mr} \cdot \frac{1}{T}$.

Setzt man endlich
$$\frac{1}{m} \cdot 2\pi\sigma^2\delta = \alpha, \quad \frac{C\delta}{mr} = \beta, \quad \frac{\sigma}{\delta} = \epsilon,$$

so ist auch
$$\rho = \frac{1}{m} \cdot \frac{4}{3}\pi\sigma^3 = \frac{2}{3}\alpha\epsilon$$

und es lässt sich die obige Zustandsgleichung in der Form schreiben:

$$p = \frac{rT}{v - \frac{2}{3}\alpha\epsilon} - \frac{arT}{(v - \frac{2}{3}\alpha\epsilon)^2} \sum_{n=1}^{n=\infty} \left(\frac{\beta}{T}\right)^{n-1} \left\{\frac{1}{n\,!} + \frac{2}{n+1\,!\,\epsilon} + \frac{2}{n+2\,!\,\epsilon^2}\right\}.$$

Die *Constanten* dieser Gleichung haben folgende Bedeutung:

Es ist α gleich dem halben im Volumen der Masseneinheit vorhandenen kritischen Raume,

$\beta r = \frac{C\delta}{m}$ gleich dem Potential der Anziehungskraft auf der Oberfläche der Deckungssphäre,

endlich $\epsilon = \frac{\sigma}{\delta}$ gleich dem Verhältnis aus dem Durchmesser des Moleküls und der Distanz, auf welche die Anziehungskraft wirkt.

Da die Gleichung 233, von welcher wir ausgegangen sind, voraussetzt, dass die Anzahl der Tripelmoleküle gegen die Anzahl der Doppelmoleküle verschwindet, so ist auch die obige Gleichung an die Voraussetzung gebunden, dass die Abweichungen des Gases vom *Boyle-Charles'schen* Gesetze noch klein sind. Es darf also auch das letzte Glied unserer Gleichung, welches ja den Innendruck darstellt, nicht über einen gewissen Wert hinaus wachsen. Dies wird um so weniger der Fall sein, je grösser ϵ ist. Aus den Versuchen von *Amagat* und *Andrews* über die Compressibilität des Kohlendioxyds berechnet sich ϵ für dieses Gas zu ungefähr 100. Nach dieser Vorstellung scheint also der Anziehungsbereich sogar noch *relativ* klein zu sein gegen den Durchmesser des Moleküls.

Wir haben bisher unsere Zustandsgleichung abgeleitet, indem wir für $f(r)$ ein bestimmtes einfaches Abhängigkeitsverhältnis einführten. Lässt man $f(r)$ ganz willkürlich, so ergibt sich leicht, dass dies den Typus der Zustandsgleichung, auf welche man kommt, in keiner Weise verändert.

Es wird stets $p = \frac{rT}{v-\rho} - \frac{A}{(v-\rho)^2}$ und es ist nur noch A von $f(r)$ abhängig.

Dies gilt freilich nur solange man die Anzahl der Tripelmoleküle und der noch höheren *Congregationen* vernachlässigen darf. Ist dies nicht mehr der Fall, so werden noch weitere Glieder hinzutreten, welche in ihren Nennern das $v - \rho$ in der dritten, vierten und höheren Potenzen enthalten. Es ergibt sich dann für p eine Potenzreihe, wie sie ähnlich auch schon Herr Professor Jäger von anderen Betrachtungen ausgehend aufgestellt hat. Leider begegnet die Auswertung ihrer weiteren *Coëfficienten* kaum zu überwindenden Schwierigkeiten.

V. *On the Solution of a Pair of Simultaneous Linear Differential Equations, which occur in the Lunar Theory.* By ERNEST W. BROWN, Sc.D., F.R.S.

[Received 1899 July 14.]

IN the calculation of the inequalities in the Moon's motion by means of rectangular coordinates a certain pair of differential equations is continually requiring solution. The left-hand members are linear and always the same; the right-hand members are known functions of the independent variable—the time—and vary with each class of inequalities considered. It has been the practice to obtain the required particular integral by assuming the solution (the form of which is known) and then to determine the coefficients by continued approximation. This method is troublesome to put into a form which a computer can use easily and is besides peculiarly liable to chance errors; a large number of processes would have to be learnt before the computer could proceed quickly and securely. The main object of this paper is to put the solution into a form which will avoid these difficulties, but I believe that some of the results may be found to be of a more general interest. Further, the question of the convergence of the series used to represent the coordinates in the Lunar Theory may be somewhat narrowed. In fact it being granted that the series forming the 'Variation' inequalities and the elliptic inequalities depending on the first power of the Moon's eccentricity are convergent, it is not difficult to demonstrate, by means of equation (14) below, that all the terms multiplied by a given combination of powers of the eccentricities, inclination and ratio of the parallaxes, that is, all the terms with a given characteristic, form a convergent series.

The equations to be considered are

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} + Lx + L'y = R,$$

$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} + L'x + L''y = R',$$

where

$$\begin{matrix} L, L'' \\ L' \end{matrix} \text{ are of the forms } \Sigma_i q_i \begin{matrix}\cos \\ \sin\end{matrix} (2i+1)(n-n')(t-t_0),$$

$$\begin{matrix} R, \\ R', \end{matrix} \text{ of the forms } \Sigma_i q_i \begin{matrix}\cos \\ \sin\end{matrix} \{i(t-t_0)+\tau(t-t_1)\}(n-n'),$$

t_0, t_1, τ, n, n', q_i being known constants, and i taking all positive and negative integral values; τ is either an integer, in which case $t_1 = t_0$, or is incommensurable with an integer.

The corresponding particular integral required is, in general,

$$\begin{matrix} x \\ y \end{matrix} = \Sigma_i \begin{matrix} p_i \cos \\ p'_i \sin \end{matrix} \{i(t-t_0)+\tau(t-t_1)\}(n-n').$$

If we substitute this solution in the differential equations and equate to zero the coefficients of like periodic terms, we obtain an infinite series of linear equations with an infinite number of unknowns. The series are assumed to be convergent and in most cases the coefficients diminish rapidly as i increases. Nevertheless, it is frequently found necessary to proceed as far as $i = \pm 5$, demanding the determination of about 20 unknowns from the same number of equations.

In the determination of the latitude the equation

$$\frac{d^2z}{dt^2} + L_1 z = R'',$$

occurs; L_1, R'' are of similar forms to L, R', respectively. If z_1, z_2 be two particular integrals of

$$\frac{d^2z}{dt^2} + L_1 z = 0,$$

it is known that the particular integral required is

$$z \,.\, C = z_2 \int z_1 R''\, dt - z_1 \int z_2 R''\, dt,$$

where C is a constant given by

$$C = z_1 \frac{dz_2}{dt} - z_2 \frac{dz_1}{dt}.$$

I shall show in what follows how we may obtain a similar expression for the solution of the simultaneous equations above, having a sufficiently simple form to be of use in computations. Later the significance of the solutions is explained and certain exceptional cases occurring in the Lunar Theory are treated. The results obtained have in fact been used in the calculation of the terms of the third* and fourth orders in relation to the eccentricities, the inclination and the ratio of the parallaxes.

* *Mem. R. A. S.*, Vol. LIII. pp. 163—202.

I.

In order that the series which occur may be all algebraical instead of trigonometrical, we use the conjugate complexes u, s, where

$$u = x + y\iota, \qquad s = x - y\iota.$$

We also put

$$\zeta = \exp.\, \iota\,(n - n')\,(t - t_0),$$

$$D = \zeta \frac{d}{d\zeta} = \frac{1}{\iota\,(n - n')}\frac{d}{dt},$$

$$\mathrm{m} = \frac{n'}{n - n'}, \quad t_0 = 0 = t_1.$$

The generality of the results is not affected by the last supposition.

The simultaneous equations then take the form

$$\left.\begin{aligned}(D + \mathrm{m})^2 + Mu + Ns &= A\\ (D - \mathrm{m})^2 + Ms + \bar{N}u &= \bar{A}\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where

$$\left.\begin{aligned}&M,\ N \text{ are of the form } \Sigma p_i \zeta^{2i},\\ &A \text{ is of the form } \Sigma p_i \zeta^{2i+1+\tau} + \Sigma p'_i \zeta^{2i+1-\tau},\end{aligned}\right\}\ i = 0,\ \pm 1, \ldots$$

$$\bar{M} = M.$$

The bar placed over a letter or expression denotes here and elsewhere that ι has been changed to $-\iota$, that is, ζ^{-1} put for ζ.

To obtain the particular integrals of equations (1), it will first be necessary to obtain four independent particular integrals of

$$\left.\begin{aligned}(D + \mathrm{m})^2 u + Mu + Ns &= 0\\ (D - \mathrm{m})^2 s + Ms + \bar{N}u &= 0\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Denote these integrals by

$$u = u_j, \quad s = s_j, \quad j = 1,\ 2,\ 3,\ 4,$$

so that if Q_j denote an arbitrary constant, the general solution of (2) is

$$u = \Sigma_j\, Q_j\, u_j, \quad s = \Sigma_j\, Q_j\, s_j, \quad j = 1,\ 2,\ 3,\ 4.$$

By supposing the Q_j to have variable instead of constant values we can then proceed to find a particular integral of (1) and thence their general solution.

In order to make certain of the later arguments clear it is necessary to indicate the manner in which the equations (1) arise.

The equations

$$D^2 u + 2\mathrm{m}\, Du + \frac{3}{2}\mathrm{m}^2 (u + s) - \frac{\kappa u}{(us)^{\frac{3}{2}}} = 0,$$

$$D^2 s - 2\mathrm{m}\, Ds + \frac{3}{2}\mathrm{m}^2 (u + s) - \frac{\kappa s}{(us)^{\frac{3}{2}}} = 0,$$

with their first integral,

$$F \equiv Du \,.\, Ds + \frac{3}{4} \mathrm{m}^2 (u+s)^2 + \frac{2\kappa}{(us)^{\frac{1}{2}}} = C,$$

admit a particular solution,

$$u = u_0 = \Sigma a_i \zeta^{2i+1}, \quad s = s_0 = \Sigma a_{-i} \zeta^{2i-1},$$

containing two arbitrary constants; these constants are the quantities denoted by n, t_0 above. The coefficients a_i are functions of n and the known constants present in the differential equations.

Put $$u = u_0 + u_1, \quad s = s_0 + s_1,$$

and, after expansion in powers of u_1, s_1, neglect squares and products of these quantities. Omitting the suffix, and giving proper meanings to M, N, the resulting equations become those denoted by (2) above.

The first integral $F = C$ becomes

$$\phi \equiv \frac{\partial F}{\partial u_0} u + \frac{\partial F}{\partial s_0} s = 0.$$

If, however, we had deduced this first integral directly from (2), it would have been $\phi = C'$, where C' is an arbitrary constant. When the equations (2) are considered independently the constant C' must be retained.

Three independent solutions of (2) are known. In finding the principal part of the motion of the lunar perigee Dr Hill* gave one of them, namely, $u = Du_0$, $s = Ds_0$, and obtained the forms of the other two; the coefficients of the latter have been obtained by myself†. It is therefore only necessary to find a fourth solution, linearly independent of the other three, in order to obtain the general solution.

II.

The Fourth Integral of the Equations.

$$(D+\mathrm{m})^2 u + Mu + Ns = 0 \quad\ldots\ldots(3),$$

$$(D-\mathrm{m})^2 s + Ms + \overline{N}u = 0 \quad\ldots\ldots(3').$$

The known integrals may be denoted by

$$\left.\begin{array}{ll} u_1 = \Sigma_i \epsilon_i \zeta^{2i+1+\mathrm{c}}, & s_1 = \Sigma_i \epsilon'_{-i} \zeta^{2i-1+\mathrm{c}} \\ u_2 = \Sigma_i \epsilon'_i \zeta^{2i+1-\mathrm{c}}, & s_2 = \Sigma_i \epsilon_{-i} \zeta^{2i-1-\mathrm{c}} \\ u_3 = \Sigma_i (2i+1) a_i \zeta^{2i+1}, & s_3 = \Sigma_i (2i-1) a_{-i} \zeta^{2i-1} \end{array}\right\} \quad\ldots\ldots(4).$$

* *Acta Math.* Vol. VIII. pp. 1—36.

† *Mem. R. A. S.* Vol. LIII. p. 94.

If Q_1, Q_2, Q_3 be three arbitrary constants, then

$$u = \Sigma_j Q_j u_j, \quad s = \Sigma_j Q_j s_j, \quad j = 1, 2, 3 \qquad (5)$$

is a solution of the equations. Owing to the introduction of Q_1, Q_2, Q_3, we can consider $u_1, \ldots s_3$ completely known; c is a constant which is supposed incommensurable with unity.

To discover the fourth integral, the method of the Variation of Arbitrary Constants is used in the usual way, by assuming that

$$u_1 DQ_1 + u_2 DQ_2 + u_3 DQ_3 = 0.$$

By substituting (4) in the differential equations we find

$$Du_1 . DQ_1 + Du_2 . DQ_2 + Du_3 . DQ_3 = 0,$$

$$\Sigma_j (s_j D^2 Q_j + 2Ds_j . DQ_j - 2\mathrm{m} s_j DQ_j) = 0 \qquad (6).$$

Put $$u_2 Du_3 - u_3 Du_2 = \alpha_1, \text{ etc.}$$

Then $$\frac{DQ_1}{\alpha_1} = \frac{DQ_2}{\alpha_2} = \frac{DQ_3}{\alpha_3} = L, \text{ suppose.}$$

Substituting in (6), the equation for L may be written,

$$(\Sigma\alpha s) DL + 2LD(\Sigma\alpha s) - L(\Sigma s D\alpha + 2\mathrm{m}\Sigma\alpha s) = 0 \qquad (6'),$$

where $$\Sigma\alpha s = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3, \text{ etc.}$$

The last term of this equation can be shown to be zero. Substitute u_1, s_1 and u_2, s_2 successively in (3): multiply the resulting equations by s_2, s_1 respectively and subtract. We thus obtain

$$(D + 2\mathrm{m})(s_2 Du_1 - s_1 Du_2) + (\mathrm{m}^2 + M)(s_2 u_1 - u_2 s_1) = 0.$$

Also, treating (3′) in a similar manner,

$$(D - 2\mathrm{m})(u_2 Ds_1 - u_1 Ds_2) + (\mathrm{m}^2 + M)(u_2 s_1 - s_2 u_1) = 0.$$

The sum of these two equations is integrable and gives

$$s_2 Du_1 - u_1 Ds_2 + u_2 Ds_1 - s_1 Du_2 + 2\mathrm{m}(s_2 u_1 - u_2 s_1) = C_{12},$$

where C_{12} is a constant. It should be noticed that this constant is not arbitrary since the values of u_1, s_1, u_2, s_2 were definitely fixed, so that C_{12} may be treated as a known constant.

Denote the last equation by

$$f_{12} = C_{12} \qquad (7).$$

We find in an exactly similar manner

$$f_{23} = C_{23}, \quad f_{31} = C_{31} \qquad (7').$$

Multiply these three equations by u_1, u_2, u_3 and add. Noticing the meanings attached to α_1, α_2, α_3, we obtain

$$u_1 C_{23} + u_2 C_{31} + u_3 C_{12} = \Sigma\alpha s.$$

Similarly $$0 = u_1 Df_{23} + u_2 Df_{31} + u_3 Df_{12}$$
$$= \Sigma s D\alpha + 2\mathrm{m}\Sigma\alpha s.$$

Substituting the last result in (6′), we find

$$\frac{DL}{L} + 2\frac{D(\Sigma \alpha s)}{\Sigma \alpha s} = 0,$$

which, on integrating, gives

$$L = \frac{L_0}{(\Sigma \alpha s)^2} = \frac{L_0}{(u_1 C_{23} + u_2 C_{31} + u_3 C_{12})^2},$$

where L_0 is a new arbitrary constant.

Thence
$$Q_1 = (Q_1) + L_0 D^{-1} \frac{\alpha_1}{(u_1 C_{23} + u_2 C_{31} + u_3 C_{12})^2}, \text{ etc.,}$$

in which (Q_1) is a new arbitrary constant and D^{-1} denotes an integration, i.e. the operation inverse to D.

If, finally, we now let Q_1, Q_2, Q_3, Q_4 represent four arbitrary constants, the general solution of (2) is

$$u = Q_1 u_1 + Q_2 u_2 + Q_3 u_3 + Q_4 u_4,$$
$$s = Q_1 s_1 + Q_2 s_2 + Q_3 s_3 + Q_4 s_4,$$

where
$$u_4 = \Sigma_j u_j D^{-1} \frac{\alpha_j}{(u_1 C_{23} + u_2 C_{31} + u_3 C_{12})^2}, \quad j = 1,\ 2,\ 3.$$

This result is true whatever particular solutions are represented by

$$u_1,\ s_1 : u_2,\ s_2 : u_3,\ s_3$$

as long as they are linearly independent. As, however, the expression for u_4 can be very much simplified by using the values given earlier, I shall immediately proceed to the special case under consideration.

It is easy to show that $C_{31} = 0 = C_{23}$. For, looking at the forms assumed, we see that u_1, s_1 contain the factor ζ^c, u_2, s_2 the factor ζ^{-c} and u_3, s_3 have no such factor. Hence f_{23} has the factor ζ^c, f_{31} the factor ζ^{-c}. As c is supposed incommensurable with unity, the equations (7′) are only possible if $C_{31} = 0$ and $C_{23} = 0$.

Hence we have

$$u_4 C_{12}^2 = u_1 D^{-1} \frac{u_2 Du_3 - u_3 Du_2}{u_3^2} + u_2 D^{-1} \frac{u_3 Du_1 - u_1 Du_3}{u_3^2} + u_3 D^{-1} \frac{u_1 Du_2 - u_2 Du_1}{u_3^2}.$$

The first two terms of the right-hand side are integrable and become

$$u_1 \frac{u_2}{u_3} - u_2 \frac{u_1}{u_3},$$

that is, zero. Whence considering C_{12}^2 as absorbed in the arbitrary Q_4, we have

$$u_4 = u_3 D^{-1} \left(\frac{u_1 Du_2 - u_2 Du_1}{u_3^2}\right) \qquad (8).$$

We may similarly show that

$$s_4 = s_3 D^{-1} \left(\frac{s_1 Ds_2 - s_2 Ds_1}{s_3^2}\right).$$

III.

Although this is probably the simplest form obtainable for u_4, it is unsuitable for calculation. The values of $u_1, \ldots$ are all of the form

sum of cosines $+ \iota$ (sum of sines).

To adapt u_4 to calculation it is best to express it in the form

$$u_3 (P + Q\iota)$$

where P, Q are real. I shall show that

$$\frac{u_4}{u_3} = D^{-1}\left(\frac{u_1 Du_2 - u_2 Du_1}{u_3^2}\right)$$

$$= \tfrac{1}{2}\frac{s_1 u_2 - u_1 s_2}{u_3 s_3} + \tfrac{1}{2}D^{-1}\left\{\frac{C_{12}}{u_3 s_3} - \frac{s_1 u_2 - u_1 s_2}{u_3 s_3}\left(2\mathrm{m} + \frac{Du_3}{u_3} - \frac{Ds_3}{s_3}\right)\right\} \quad \ldots\ldots\ldots\ldots\ldots(9).$$

Since $f_{23} = 0 = f_{31}$ and $f_{12} = C_{12}$, we have

$$-\tfrac{1}{2}\frac{C_{12}}{u_3 s_3} = \frac{u_2 f_{13} - u_1 f_{23}}{u_3^2 s_3} - \tfrac{1}{2}\frac{f_{12}}{u_3 s_3} = -\frac{u_1 Du_2 - u_2 Du_1}{u_3^2} + \frac{u_2 Ds_1 - u_1 Ds_2 + s_1 Du_2 - s_2 Du_1}{2u_3 s_3}$$

$$-\frac{s_1 u_2 - u_1 s_2}{u_3 s_3}\frac{Du_3}{u_3} - \mathrm{m}\frac{s_1 u_2 - u_1 s_2}{u_3 s_3} = -\frac{u_1 Du_2 - u_2 Du_1}{u_3^2} + \tfrac{1}{2}D\left(\frac{s_1 u_2 - u_1 s_2}{u_3 s_3}\right)$$

$$-\frac{s_1 u_2 - u_1 s_2}{u_3 s_3}\left(\frac{Du_3}{u_3} - \frac{Ds_3}{s_3}\right) - \mathrm{m}\frac{s_1 u_2 - u_1 s_2}{u_3 s_3}.$$

Submitting this to the operation D^{-1} and transposing we obtain the required expression.

It is easy to see that (9) is of the required form. For when we put $-\iota$ for ι, that is, ζ^{-1} for ζ, the expressions

$$u_1,\ u_2,\ s_1,\ s_2,\ u_3,\ s_3,\ D^{-1},\ D \text{ respectively}$$

become
$$s_2,\ s_1,\ u_2,\ u_1,\ -s_3,\ -u_3,\ -D^{-1},\ -D;$$

the first term of (9) is therefore unchanged, while the second term simply changes sign. Hence the first term is real and the second a pure imaginary.

IV.

It is necessary to examine the four solutions and especially the one last found a little more closely. Write

$$u_4 = u_3 (P + D^{-1}P_1).$$

The expressions (4) show that P and P_1, being both real, will be expressible as sums of cosines of multiples of the angle $2(n - n')t$. As P_1 contains a constant term B, $D^{-1}P_1$ contains a term of the form $\iota Bt(n - n')$, and therefore u_4 is of the form

$$u_3\{\iota Bt(n - n') + \text{a power series in } \zeta^2\}.$$

It is therefore of the same form as u_3, except for the part

$$\iota Btu_3 (n - n').$$

We saw earlier that the equations (2) admit of a first integral

$$\phi = C',$$

and that this should be derivable from the integral

$$F = C,$$

of the non-linear equations when the former are considered as derived from the latter. The constant C' should therefore in this case be zero. It is easy to see that the constant is zero when we substitute in ϕ the solutions u_1, s_1 or u_2, s_2 or u_3, s_3. For the solution u_4, s_4, the constant takes the value C_{12} which is not zero. Hence though (u_4, s_4) belongs to the linear equations (2) it plays no part in the non-linear equations from which these were derived.

The solutions u_1, s_1 and u_2, s_2 are those used in developing the Lunar Theory; they contain the terms dependent on the first power of the lunar eccentricity. It is necessary to see why the solutions u_3, s_3 and u_4, s_4 are not used in the development.

The particular solution of the original equations of which use was made was

$$u = u_0, \quad s = s_0,$$

where $$u_0 = \Sigma_i a_i \zeta^{2i+1} = \Sigma_i a_i \text{ exp. } (2i+1)(n-n')(t-t_0).$$

If we add a small quantity δt_0 to t_0 (which is an arbitrary constant of this solution) the resulting expression will still be a solution. Expand in powers of δt_0 neglecting squares and higher powers. The additions to u_0, s_0 will be

$$\delta u = \frac{\partial u_0}{\partial t_0}\delta t_0 = -Du_0 . \delta t_0, \quad \delta s = \frac{\partial s_0}{\partial t_0}\delta t_0 = -Ds_0 . \delta t_0.$$

These values when substituted for u, s in (2) must satisfy them independently of the value of δt_0. Hence $u = kDu$, $s = kDs$ is a solution obtained merely by altering the arbitrary t_0 and is therefore unnecessary for the development of the Lunar Theory.

The other arbitrary constant in u_0 is n, and the coefficients a_i are functions of n. If we make a small addition δn to n and proceed as before we see that

$$u = k\frac{\partial u_0}{\partial n}, \quad s = k\frac{\partial s_0}{\partial n}$$

is a solution of the linear equations (2). It is only necessary to identify this with u_4, s_4.

The forms for both are evidently the same. For we have

$$\frac{\partial u_0}{\partial n} = \Sigma_i \left\{\frac{\partial a_i}{\partial n} + (2i+1)(t-t_0)a_i\right\} \text{ exp. } (2i+1)(n-n')(t-t_0)$$

$$= \Sigma_i \frac{\partial a_i}{\partial n} \text{ exp. } (2i+1)(n-n')(t-t_0) + (t-t_0)Du_0.$$

The terms with t as factor agree (t_0 was put zero in the expression for u_4) when the proper constant factor is introduced, and the remaining parts are of the same form. As no linear relation can exist between the first three solutions and either of the forms for the fourth solution, these two forms must be the same except as to a constant factor. Hence

$$k\frac{\partial u_0}{\partial n} = u_3 D^{-1}\left(\frac{u_1 D u_2 - u_2 D u_1}{u_3^2}\right).$$

This relation is a somewhat remarkable one. In investigations where the arbitrary constants are varied—and there are many such—we have a means of obtaining $\frac{\partial x}{\partial n}$, $\frac{\partial y}{\partial n}$ (which are the most troublesome to find) when the numerical value of the ratio n'/n has been used in finding x, y. A direct proof of this relation is desirable. This and the theorems which I have given elsewhere* are probably particular cases of some much more general theorem. Thus, of the four integrals of the linear equations two only are required for the development of the lunar theory, the other two arising from additions to the arbitrary constants in the particular solution of the original equations.

V.

Having obtained the solution of

$$(D+\mathrm{m})^2 u + Mu + Ns = 0,$$

$$(D-\mathrm{m})^2 s + Ms + \overline{N}u = 0,$$

in the form $$u = \Sigma Q_j u_j, \quad s = \Sigma Q_j s_j, \quad j = 1, 2, 3, 4,$$

the next problem is to find the solution of

$$(D+\mathrm{m})^2 u + Mu + Ns = A,$$

$$(D-\mathrm{m})^2 s + Ms + \overline{N}u = \overline{A},$$

where A, $\overline{A}$ are functions of the time.

Following the usual method of varying the arbitraries we have

$$\left.\begin{aligned} \Sigma Du_j . DQ_j = A, \quad & \Sigma Ds_j . DQ_j = \overline{A} \\ \Sigma u_j DQ_j = 0, \quad & \Sigma s_j DQ_j = 0 \end{aligned}\right\} \qquad (10).$$

These must be solved in order to find the variable values of the arbitraries. The only difficulty is to find these values in forms sufficiently simple to be of use.

The expressions at the end of II. show that we can derive s_4/s_3 from u_4/u_3 by putting ζ^{-1} for ζ and changing the sign. For u_1, s_2 interchange as do u_2, s_1, while D changes sign: u_3 becomes $-s_3$. Since

$$u_4 = u_3(P + Q\iota),$$

we have $$s_4 = s_3(-P + Q\iota).$$

Hence $$u_4 s_3 - s_4 u_3 = 2u_3 s_3 P$$

$$= u_2 s_1 - u_1 s_2 \qquad (11)$$

by the result obtained in III.

* *Proc. London Math. Soc.* Vol. XXVIII. pp. 143—155.

Again, as the first integral obtained in II. is equally applicable to u_4, s_4, we have

$$C_{34} = f_{34} = s_4 Du_3 + u_4 Ds_3 - u_3 Ds_4 - s_3 Du_4 + 2\mathrm{m}\,(s_4 u_3 - u_4 s_3),$$

which, by inserting the expressions for u_4, s_4 just given, becomes

$$C_{34} = -2\,(s_3 Du_3 - u_3 Ds_3)\,P - 2\iota u_3 s_3 DQ + 2\mathrm{m}\,(s_2 u_1 - u_2 s_1),$$

or, using the values of P, Q obtained in III.,

$$C_{34} = -(s_3 Du_3 - u_3 Ds_3)\frac{s_1 u_2 - u_1 s_2}{u_3 s_3} - C_{12}$$
$$+ (s_1 u_2 - u_1 s_2)\left(2\mathrm{m} + \frac{Du_3}{u_3} - \frac{Ds_3}{s_3}\right) + 2\mathrm{m}\,(s_2 u_1 - s_1 u_2),$$

whence $$C_{34} = -C_{12} \quad\text{.......................................(12).}$$

We can show as in II. that $C_{14} = 0 = C_{24}$.

Solving equations (10) we obtain

$$DQ_j = \frac{\Delta_j}{\Delta},$$

where

$$\Delta = \begin{vmatrix} Du_1, & Du_2, & Du_3, & Du_4 \\ Ds_1, & Ds_2, & Ds_3, & Ds_4 \\ u_1, & u_2, & u_3, & u_4 \\ s_1, & s_2, & s_3, & s_4 \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} A, & Du_2, & Du_3, & Du_4 \\ \overline{A}, & Ds_2, & Ds_3, & Ds_4 \\ 0, & u_2, & u_3, & u_4 \\ 0, & s_2, & s_3, & s_4 \end{vmatrix}, \text{ etc.}$$

In the determinant Δ the first minor of Du_1 is

$$Ds_2\,(u_3 s_4 - s_3 u_4) + Ds_3\,(u_4 s_2 - s_4 u_2) + Ds_4\,(u_2 s_3 - s_2 u_3),$$
$$= s_2 f_{34} + s_3 f_{42} + s_4 f_{23},$$
$$= s_2 C_{34} + s_3 C_{42} + s_4 C_{23}.$$

Also, the first minor of Ds_1 is similarly

$$-(u_2 C_{34} + u_3 C_{42} + u_4 C_{23}).$$

The other minors of the elements in the first two rows of Δ are similar, the suffixes following a cyclical order. We have thus all the minors of the elements A, $\overline{A}$ in the determinants Δ_j.

Remembering that $C_{34} = -C_{12}$ and that all the other constants C_{ij} are zero, we obtain

$$\Delta_1 = -(s_2 A + u_2 \overline{A})\,C_{12},$$
$$\Delta_2 = (s_1 A + u_1 \overline{A})\,C_{12},$$
$$\Delta_3 = (s_4 A + u_4 \overline{A})\,C_{12},$$
$$\Delta_4 = -(s_3 A + u_3 \overline{A})\,C_{12},$$

and $$\Delta = -(s_2 Du_1 - s_1 Du_2 - s_4 Du_3 + s_3 Du_4)\,C_{12}.$$

But the effect of putting ζ^{-1} for ζ in Δ is only to interchange an even number of rows and columns and therefore to leave Δ unaltered. Making this change in the last equation we find

$$\Delta = -(-u_1 Ds_2 + u_2 Ds_1 - u_4 Ds_3 + s_4 Du_3)\, C_{12}.$$

Whence, by addition,

$$\begin{aligned} 2\Delta &= -\{f_{12} - 2\mathrm{m}(s_2 u_1 - u_2 s_1) - f_{34} + 2\mathrm{m}(s_4 u_3 - u_4 s_3)\}\, C_{12} \\ &= -(C_{12} - C_{34})\, C_{12} = -2C_{12}^2, \end{aligned}$$

in virtue of (7) and (12). Hence $\Delta = -C_{12}^2$.

Finally,
$$\Delta Q_1 = \frac{\Delta_1}{\Delta} = \frac{s_2 A + u_2 \bar{A}}{C_{12}}, \text{ etc.}$$

and
$$Q_1 = \frac{1}{C_{12}} D^{-1}(s_2 A + u_2 \bar{A}), \text{ etc.}$$

And the particular integral corresponding to the right-hand members, A, $\bar{A}$, is

$$\begin{aligned} u = \frac{1}{C_{12}} \{&u_1 D^{-1}(s_2 A + u_2 \bar{A}) - u_2 D^{-1}(s_1 A + u_1 \bar{A}) \\ &- u_3 D^{-1}(s_4 A + u_4 \bar{A}) + u_4 D^{-1}(s_3 A + u_3 \bar{A})\} \quad \ldots\ldots(13), \\ s = \frac{1}{C_{12}} \{&s_1 D^{-1}(s_2 A + u_2 \bar{A}) - s_2 D^{-1}(s_1 A + u_1 \bar{A}) \\ &- s_3 D^{-1}(s_4 A + u_4 \bar{A}) + s_4 D^{-1}(s_3 A + u_3 \bar{A})\}. \end{aligned}$$

It is easy to see that s is derivable from u (as it should be) by putting ζ^{-1} for ζ. In fact, the coefficient of u_1 in the first term is conjugate to that of u_2 in the second term, that of u_3 in the third term is a pure imaginary and that of u_4 in the last term is real.

VI.

In the applications of this result to the Lunar Theory A is always an expression of the form

$$\Sigma_i q_i \zeta^{i+\tau} + \Sigma_i q_i' \zeta^{i-\tau}, \quad i = 0, \pm 1, \pm 2, \ldots,$$

where τ, q_i, q_i' are known constants; $\bar{A}$ is derived from A by putting ζ^{-1} for ζ. Thus A, $\bar{A}$ are conjugate complexes whose real and imaginary parts are respectively sums of cosines and sines. The corresponding particular integral should in general be of the same form. Hence a difficulty arises owing to the fact that u_4, s_4 contain t in a non-periodic form. I shall now show that in general all the non-periodic parts disappear from the particular integral.

Put

$$\begin{aligned} u_4 &= u_4' + \iota B u_3 t (n - n'), \\ s_4 &= s_4' + \iota B s_3 t (n - n'). \end{aligned}$$

Then u_4', s_4' are periodic. The sum of the third and fourth terms of (13) becomes

$$-u_3D^{-1}(s_4'A+u_4'\bar{A})+u_4'D^{-1}(s_3A+u_3\bar{A})$$
$$-[u_3D^{-1}\{(s_3A+u_3\bar{A})t\}+u_3tD^{-1}(s_3A+u_3\bar{A})]\iota B(n-n').$$

The first line of this expression is in general periodic. The second line becomes, on integrating its first term by parts,

$$u_3BD^{-2}(s_3A+u_3\bar{A}).$$

The non-periodic part thus disappears.

When we perform the double integration involved in this last expression, we obtain

$$u_3\{C_0+C_1\iota(n-n')t+\text{periodic part}\}$$

where C_0, C_1 are arbitraries. The terms containing C_0, C_1 are simply parts of the complementary function and may be considered as contained in $Q_3u_3+Q_4u_4$. The particular integral may therefore be written

$$u=\frac{1}{C_{12}}[u_1D^{-1}(s_2A+u_2\bar{A})-u_2D^{-1}(s_1A+u_1\bar{A})+u_4'D^{-1}(s_3A+u_3\bar{A})$$
$$-u_3D^{-1}\{s_4'A+u_4'\bar{A}-BD^{-1}(s_3A+u_3\bar{A})\}]\text{......(14)},$$

which is its final form.

VII.

In general this particular integral consists only of periodic terms. There are, however, two cases in which non-periodic terms may arise. If $\tau=$ an odd integer, that is, if A is of the form $\Sigma q_i\zeta^{2i+1}$, the integrals multiplied by u_4' and u_4 might give rise to terms of the form αt where α is a constant.

In this case, $s_3A+u_3\bar{A}$ is of the form $\Sigma\beta_i(\zeta^{2i}-\zeta^{-2i})$ and therefore its integral will be periodic. The last term of (14) is of the form

$$-u_2D^{-1}(\text{const.}+\text{power series in }\zeta^2),$$
$$=-u_3(tk+k'+\text{power series in }\zeta^2),$$

k, k' being constants, the former definite and the latter arbitrary. The terms $-u_3(tk+k')$ may be written

$$-k'u_3-\{u_3tB\iota(n-n')+u_4'\}\frac{k}{B\iota(n-n')}+\frac{ku_4'}{B\iota(n-n')}.$$

The first two terms of this may be considered as included in the part $Q_3u_3+Q_4u_4$ of the complementary function; the last part is definite and periodic. Hence no non-periodic part remains.

The second case of non-periodicity occurs when

$$A=\Sigma_iq_i\zeta^{2i+1+c}+\Sigma_iq_i'\zeta^{2i+1-c}.$$

Here the first two terms of (14) may give rise to the non-periodic part

$$\{u_1t\iota(n-n')[s_2A+u_2\bar{A}]_0-u_2t\iota(n-n')[s_1A+u_1\bar{A}]_0\}\div C_{12},$$