The equation of a circle cutting $z\bar{z}+K=0$ orthogonally is $z\bar{z}+\bar{p}z+p\bar{z}-K=0$.

Let $C(-p)$ be the centre, $P(z)$ and $P'(z')$ a pair of inverse points.

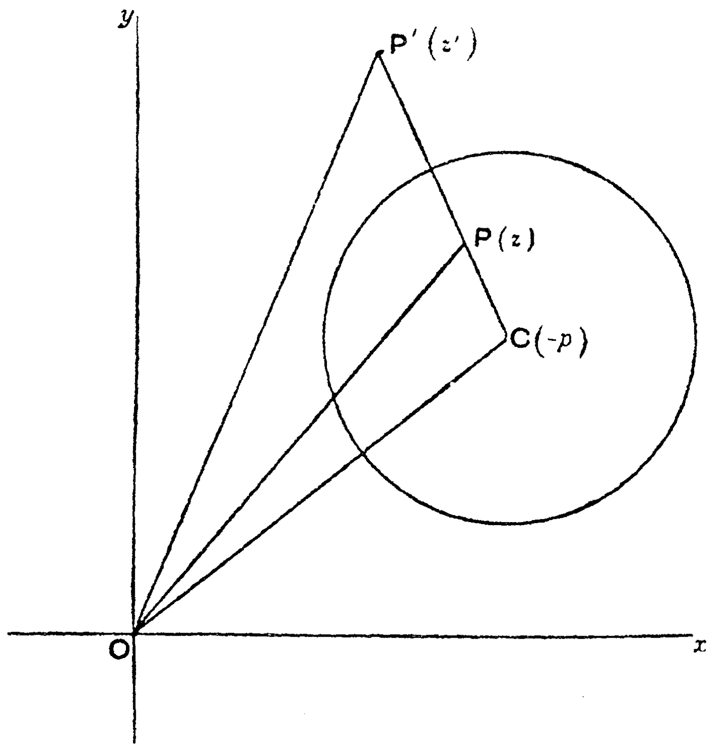

FIG. 99.

Let the complex numbers represented by CP and CP' be u, u'. Then

$$z=-p+u, \qquad z'=-p+u'.$$

Also, since u, u' have the same amplitude, and the product of their moduli is equal to the square of the radius of the circle of inversion,

$$u\bar{u}'=p\bar{p}+K.$$

Therefore $$(z+p)(\bar{z}'+\bar{p})-p\bar{p}-K=0,$$

or $$z\bar{z}'+\bar{p}z+p\bar{z}'-K=0,$$

i.e. $$z=\frac{-p\bar{z}'+K}{\bar{z}'+\bar{p}}.$$

A second inversion in the circle $zz + qz + q\bar{z} - K = 0$ gives

$$z = \frac{(K + p\bar{q})z'' - K(p - q)}{(\bar{p} - \bar{q})z'' + (K + \bar{p}q)}.$$

This will not hold when the circle of inversion is a straight line $\theta = \phi$. Here inversion becomes reflexion, and we have

$$z = re^{i\theta}, \quad z' = re^{i(2\phi - \theta)}, \quad \bar{z}' = re^{i(\theta - 2\phi)};$$

therefore $$z = \bar{z}'e^{2i\phi}.$$

This combined with an inversion gives

$$z = \frac{-\bar{p}z'' + K}{z'' + p} e^{2i\phi}.$$

Let $\phi = \frac{\pi}{2} - \psi$, $\beta = e^{i\psi}$; then $e^{2i\phi} = -e^{-2i\psi} = -\frac{\bar{\beta}}{\beta}$. Then, if $p\beta = \alpha$, the transformation becomes

$$z = \frac{\bar{\alpha}z'' - K\bar{\beta}}{\beta z'' + \alpha}.$$

Hence in either case the transformation is of this form. Hence *the general displacement of a plane figure is equivalent to a pair of inversions in two circles which cut the fundamental circle orthogonally.*

33. Types of motions.

In the general displacement there are always two points which are unaltered, for if $z' = z$ we have the quadratic equation

$$\beta z^2 + (\alpha - \bar{\alpha})z + K\bar{\beta} = 0.$$

If we substitute $z = -K/\bar{z}'$, the equation becomes

$$\bar{\beta}\bar{z}'^2 + (\bar{\alpha} - \alpha)\bar{z}' + K\beta = 0;$$

therefore z' is also a root. The two points are therefore inverses with regard to the fundamental circle. This point-pair corresponds to the centre of rotation in the general displacement. In hyperbolic geometry there are three distinct types of displacement according as the centre of rotation is real, ideal, or at infinity. The first case is similar to ordinary rotation; the second case is motion of translation along a fixed line, and points not on this line describe equidistant-curves; in the third case all points describe arcs of horocycles.

34. The distance-function.

We have now to find the expression for the distance between two points P, Q, *i.e.* the function of their co-ordinates or complex numbers (z_1, z_2), which remains invariant during a motion.

The two points determine uniquely a circle cutting the fundamental circle orthogonally in X, Y. This circle represents the straight line joining PQ, and X, Y represent the points at infinity on this line. If the motion is one of translation along this line, the straight line as well as the fundamental circle are unaltered, and X, Y are fixed points. Let x, y be the complex numbers corresponding to X, Y; then the cross-ratio $(z_1 z_2, xy)$ remains constant. If we suppose, therefore, that for points on this line the distance (PQ) is a function of this cross-ratio, we can write $(PQ) = f(z_1, z_2)$. If P, Q, R are three points on the line, corresponding to the numbers z_1, z_2, z_3, this function has to satisfy the relation $(PQ) + (QR) = (PR)$, or

$$f(z_1, z_2) + f(z_2, z_3) = f(z_1, z_3).$$

This is a functional equation by which the form of the function is determined. Consider z as a parameter determining the position of a point, and differentiate with regard to z_1. Then, since

$$(z_1 z_2, xy) = \frac{z_1 - x}{z_1 - y} \cdot \frac{z_2 - y}{z_2 - x} = \frac{PX}{PY} \cdot \frac{QY}{QX},$$

we have

$$f'(z_1, z_2)\frac{QY}{QX}\frac{\partial}{\partial z_1}\left(\frac{PX}{PY}\right)=f'(z_1, z_3)\frac{RY}{RX}\frac{\partial}{\partial z_1}\left(\frac{PX}{PY}\right).$$

Hence

$$\frac{f'(z_1, z_2)}{f'(z_1, z_3)}=\frac{QX}{QY}\cdot\frac{RY}{RX}=\left(\frac{PX}{PY}\cdot\frac{RY}{RX}\right)\div\left(\frac{PX}{PY}\cdot\frac{QY}{QX}\right)=\frac{(z_1z_3,\ xy)}{(z_1z_2,\ xy)},$$

i.e. $\quad (z_1, z_2)f'(z_1, z_2)=(z_1, z_3)f'(z_1, z_3)=\text{const.}=\mu.$

Integrating, we find

$$f(z_1, z_2)=\mu\log(z_1z_2,\ xy)+C,$$

and substituting in the functional equation we find $C=0$.

Hence

$$(PQ)=\mu\log(z_1z_2,\ xy)=\mu\log\left(\frac{PX}{PY}\cdot\frac{QY}{QX}\right)=\mu\log(PQ,\ XY),$$

(PQ, XY) being the cross-ratio of the four points P, Q, X, Y on the circle, *i.e.* the cross-ratio of the pencil $O(PQ, XY)$, where O is any point on the circle.

In hyperbolic geometry $K=-k^2$, and the fundamental circle is real. The distance between two conjugate points is $\frac{1}{2}i\pi k$, and the cross-ratio $(PQ, XY)=-1$. Then

$$(PQ)=\mu i\frac{\pi}{2}.$$

Therefore $\mu=k$.

35. The line-element.

If, returning to the stereographic projection, we take the formulae in § 22, we can find an expression for the line-element ds. We have, x, y, z being the coordinates of a point on the sphere,

$$ds^2=dx^2+dy^2+dz^2.$$

Expressing this in terms of x' and y', we get

$$ds^2=\frac{4k^2d^2(dx'^2+dy'^2)}{(x'^2+y'^2+d^2)^2}.$$

In particular, if $d=2k$, so that the plane of projection is the tangent plane at A (Fig. 92), we get

$$ds = \sqrt{dx'^2+dy'^2}/\{1+\tfrac{1}{4}a(x'^2+y'^2)\},$$

where $a=1/k^2$.

36. There is a gain in simplicity when the fundamental circle is taken as a straight line, say the axis of x. Then straight lines are represented by circles with their centres on the axis of x. Pairs of points equidistant from the axis

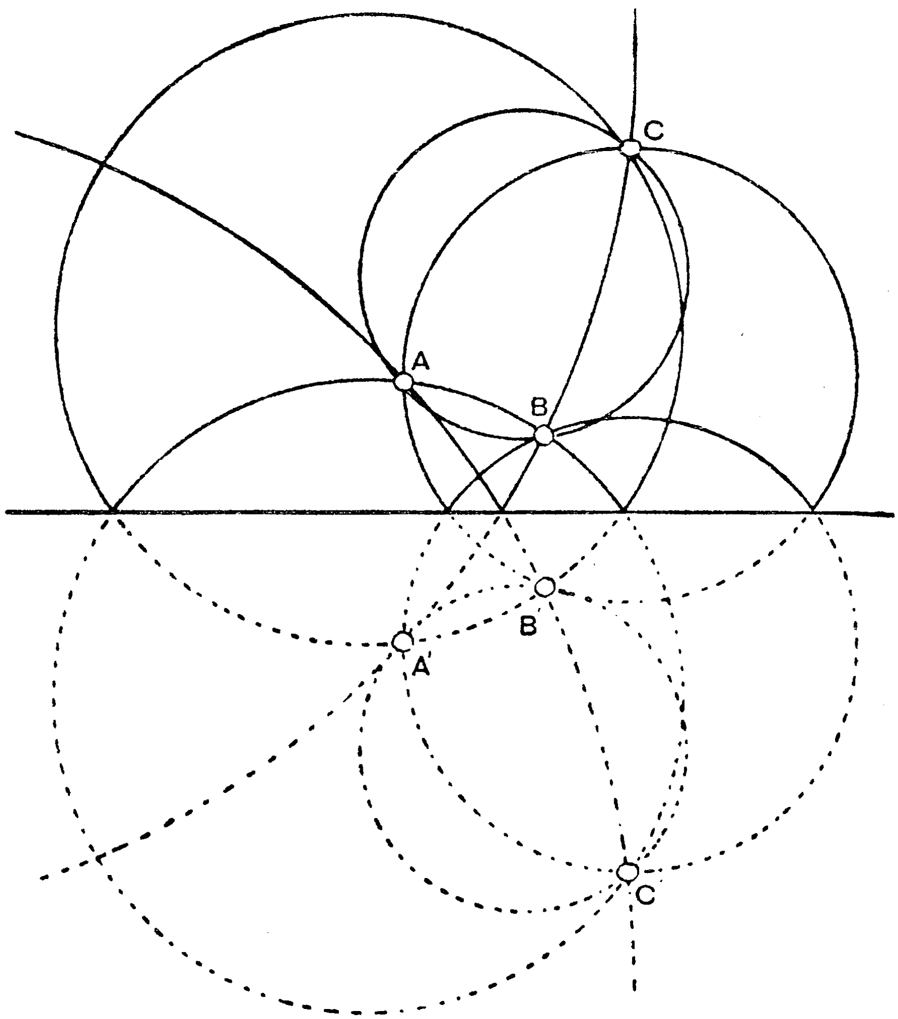

FIG. 100.

of x represent the same point, and we may avoid dealing with pairs of points by considering only those points above the x-axis. A proper circle is represented by a circle lying entirely above the x-axis ; a horocycle by a circle touching the x-axis ; an equidistant-curve by the upper part of a circle cutting the x-axis together with the reflexion of the part which lies below the axis.

Through three points A, B, C pass four circles. If A', B', C' are the reflexions of A, B, C, the four circles are represented by ABC, $A'BC$, $AB'C$, ABC'. The last three are certainly equidistant-curves ; the first may be a proper circle, a horocycle or an equidistant-curve.

37. Angle at which an equidistant-curve meets its axis.

Fig. 100 shows that the two branches of an equidistant-curve cut at infinity at a finite angle, a fact that is not apparent in the Cayley-

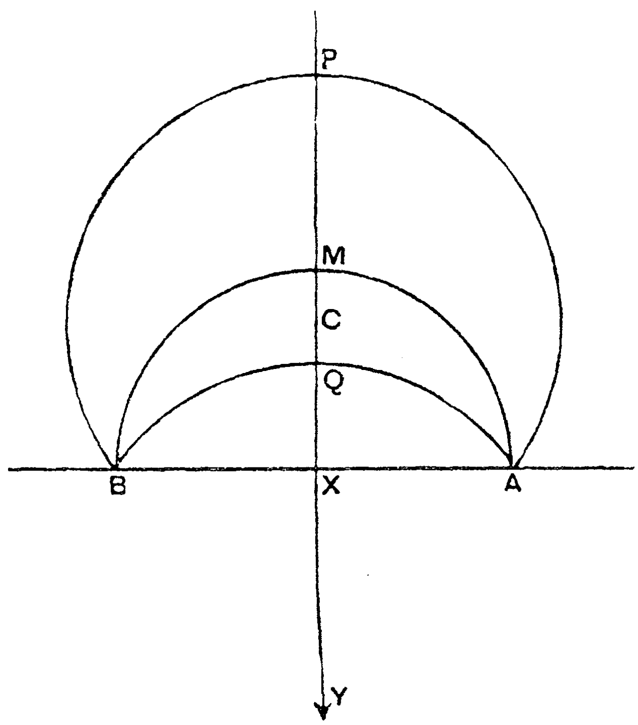

FIG. 101.

Klein representation. Let $APBQA$ (Fig. 101) be the equidistant-curve, AMB its axis, represented by the circle on AB as diameter, and let C be the centre of the circle APB. Draw $CX \perp AB$ meeting the two branches of the equidistant-curve and its axis in P, Q, M.

Let $PAQ = 2\alpha$; then $CAX = \alpha$, $\tan\alpha = \frac{CX}{XA}$. Let d be the distance of the equidistant-curve from its axis.

The line PX being $\perp AB$ represents a straight line; it cuts AB in X, and the second point at infinity on the line is represented by Y at infinity.

Hence $d = k\log(PM, XY) = k\log\frac{PX}{MX}$, also $d = k\log\frac{MX}{QX}$.

Now $$CX = \tfrac{1}{2}(PX - QX);$$

therefore $$\tan\alpha = \frac{PX - QX}{2MX} = \frac{1}{2}\left(e^{\frac{d}{k}} - e^{-\frac{d}{k}}\right) = \sinh\frac{d}{k}.$$

We can get a geometrical meaning for this result. Draw $PL \perp PN$ and $PE \parallel NE$ (Fig. 102). Then the equidistant-curve and the parallel and the axis all meet at infinity at E.

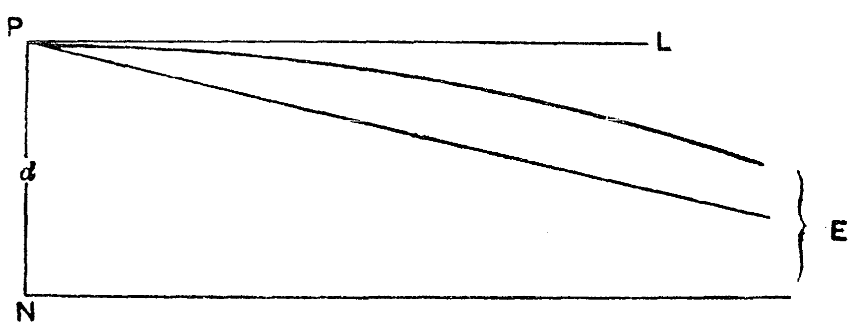

FIG. 102.

The angle LPE is then $= \alpha$.

Consider a chord PQ of the equidistant-curve: like a circle, the curve cuts the chord at equal angles. Keeping P fixed, let Q go to infinity. PQ becomes parallel to NE, and makes a zero angle with it; hence the angle between the curve and the axis is equal to the angle LPE.

The explanation of the apparent contradiction shown in the Cayley-Klein representation, where the two branches of the equidistant-curve form one continuous curve, lies in the fact that the angle between two lines becomes indeterminate when their point of intersection is on the absolute and at the same time one of the lines touches the absolute. If the first alone happens the angle is zero, if the second the angle is infinite.

38. Extension to three dimensions.

The conformal representation of non-euclidean geometry can be extended to three dimensions, planes being represented by spheres cutting a fundamental sphere orthogonally. A proper sphere is represented by a sphere which does not cut the fundamental sphere, a horosphere by a sphere touching the fundamental sphere, and an equidistant-surface by a sphere cutting the fundamental sphere.

A horocycle is represented by a circle touching the fundamental sphere. The horocycles which lie on a horosphere all pass through the same point on the sphere, viz. the point of contact. This is exactly similar to the system of circles on a plane representing the straight lines of euclidean geometry, and thus we have another verification that the geometry on the horosphere is euclidean.

This suggests that the three geometries can be represented on the plane of any one of them by systems of circles cutting a fixed circle orthogonally.

CHAPTER VI.

"SPACE CURVATURE" AND THE PHILOSOPHICAL BEARING OF NON-EUCLIDEAN GEOMETRY.

1. Four periods in the history of non-euclidean geometry.

The projective and the geodesic representations of non-euclidean geometry have an important bearing on the history of the subject, for it was through these that Cayley and Riemann arrived independently at non-euclidean geometry.

Klein has divided the history of non-euclidean geometry into three periods. The *first period*, which contains GAUSS, LOBACHEVSKY and BOLYAI, is characterised by the synthetic method, and applies the methods of elementary geometry. The *second period* is related to the geodesic representation, and employs the methods of differential geometry. It begins with RIEMANN'S classical dissertation, and includes also the work of HELMHOLTZ, LIE and BELTRAMI on the formula for the line-element. The *third period* is related to the projective representation, and applies the principles of pure projective geometry. It begins with CAYLEY, whose ideas were developed and put into relationship with non-euclidean geometry by KLEIN. To these a *fourth period* has now to be added, which is connected with no representation, but is concerned with the logical grounding of geometry upon sets of axioms. It is inaugurated by PASCH, though we must go back to VON STAUDT for the true beginnings. This period contains HILBERT and an Italian school represented by PEANO and PIERI; in America its chief representative is VEBLEN. It has led to the severe logical examination of the foundations of mathematics represented by the *Principia Mathematica* of RUSSELL and WHITEHEAD.

2. "Curved space."

If we attempt to extend the geodesic representation of non-euclidean geometry to space of three dimensions, we find ourselves at a loss, for the representation of plane geometry already requires three dimensions. It is quite a legitimate mathematical conception, however, to extend space to four dimensions. A limited portion of elliptic space of three dimensions could be represented on a portion of a "hypersphere" in space of four dimensions, or the whole of elliptic space of three dimensions could be represented completely on a hypersphere, with the understanding that a point in elliptic space is represented by a pair of antipodal points on the hypersphere.

A hypersphere is a locus of constant curvature, just as a sphere is a surface of constant curvature. Analogy with the geometry of surfaces leads to the conception of the curvature of a three-dimensional locus in space of four dimensions, and just as the curvature of a surface can be determined at any point by intrinsic considerations, such as by measuring the angles of a geodesic triangle, so by similar measurements in the three-dimensional locus we could, without going outside that locus, obtain a notion of its curvature.

3. Application of differential geometry.

This was the path traversed by RIEMANN in his celebrated Dissertation. Space, he teaches us, is an example of a "manifold" of three dimensions, distinguished from other manifolds by nature of its homogeneity and the possibility of measurement. Space is unbounded, but not necessarily infinite. Thereby he expresses the possibility that the straight line may be of finite length, though without end—a conception that was absent from the minds of any of his predecessors. The position of a point P can be determined by three numbers or coordinates, x, y, z; and if $x+dx$, $y+dy$, $z+dz$ are the values of the coordinates for a neighbouring point Q, then the length of the small element of length PQ, $=ds$, must be expressed in terms of the increments dx, dy, dz. If the increments are all increased in the same ratio, ds will be increased in the same ratio, and if all the increments are changed in sign the value of ds will be unaltered. Hence ds must be an even root, square, fourth, etc., of a positive homogeneous function of dx, dy, dz of the second, fourth, etc., degree. The simplest hypothesis is that ds^2 is a homogeneous function of dx, dy, dz of the second degree, or by proper choice of coordinates $ds^2=$ a homogeneous linear expression in dx^2, dy^2, dz^2. For example, with rectangular coordinates in ordinary space, $ds^2=dx^2+dy^2+dz^2$.

By taking the analogy of Gauss' formulae for the curvature of a surface, Riemann defines a certain function of the differentials as the measure of curvature of the manifold. In order that congruence of figures may be possible, it is necessary that the measure of curvature should be everywhere the same; but it may be positive or zero. (Riemann had no conception of Lobachevsky's geometry, for which the measure of curvature is negative.) He gives without proof the following expression for the line-element. If a denotes the measure of curvature, then

$$ds = \sqrt{\Sigma dx^2}/(1 + \tfrac{1}{4}a\Sigma x^2).$$

(Cf. Chap. V. § 35.) If k is what has already been called the space-constant, $a = 1/k^2$.

4. Free mobility of rigid bodies.

About the same time that Riemann's Dissertation was being published, Hermann von HELMHOLTZ (1821-1894) was conducting very similar investigations from the point of view of the general intuition of space, being incited thereto by his interest in the physiological problem of the localisation of objects in the field of vision.

Helmholtz [1] starts from the idea of congruence, and, by assuming certain principles such as that of *free mobility of rigid bodies*, and *monodromy*, *i.e.* that a body returns unchanged to its original position after rotation about an axis, he proves—what is arbitrary in Riemann's investigation—that the square of the line-element is a homogeneous function of the second degree in the differentials.

That the form of the function which expresses the distance between two points is limited by the possibility of the existence of congruent figures in different positions is shown as follows. Suppose we have five points in space, A, B, C, D, E. The position of each point is determined by three coordinates, and connecting each pair of points there is a certain expression involving the coordinates, which corresponds to the distance between the two points. Let

[1] "Ueber die Thatsachen, die der Geometrie zum Grunde liegen," *Göttinger Nachrichten*, 1868. An abstract of this paper was published in 1866.

us try to construct a figure $A'B'C'D'E'$ with exactly the same distances between pairs of corresponding points as the figure $ABCDE$. A' may be taken arbitrarily. Then B' must lie on a certain surface, since its coordinates are connected by one equation. C' has to satisfy two conditions, and therefore lies on some curve, and then D' is completely determined by its distances from A', B' and C'. Similarly E' will be completely determined by its distances from A', B' and C', but we cannot now guarantee that the distance $D'E'$ will be equal to DE. The distance-function is thus limited by one condition. And with more than five points a still greater number of conditions must be satisfied.[1]

It is customary to speak, as Helmholtz does, of the transformation of a figure into another congruent figure as a *displacement* of a single *rigid* figure from one position to another. This language often enables us to abbreviate our statements.

Thus, employing this language, we may argue for the general case as follows. If there are n points, the figure has $3n$ degrees of freedom, and there are $\frac{1}{2}n(n-1)$ equations connecting the distances of pairs of points. But a rigid body has only 6 degrees of freedom; therefore the number of equations determining the distance-function is $\frac{1}{2}n(n-1)-3n+6=\frac{1}{2}(n-3)(n-4)$.

But it is necessary to avoid here a dangerous confusion. Points in space are fixed objects and cannot be conceived as altering their positions. When we speak of a motion of a rigid figure we are thinking of material bodies. The assumption which Helmholtz makes, which is expressed by the phrase, the "free mobility of rigid bodies," is thus

[1] This method was employed by J. M. de Tilly, *Bruxelles, Mém. Acad. Roy.* (8vo collection), **47** (1893), to find the expression for the distance-function without using infinitesimals.

simply an assumption that there is such a thing as absolute space.

While, psychologically, the idea of congruence may be based on the idea of rigid bodies, if it were really dependent upon the actual existence of rigid bodies it would have a very insecure foundation. Not only are the most solid bodies within our experience elastic and deformable, but modern researches in physics have given a high degree of probability to the conception that all bodies suffer a change in their dimensions when they are in motion relative to the aether. As all bodies, including our measuring rods, suffer equally in this distortion, however, we can never be conscious of it.

5. Continuous groups of transformations.

Helmholtz's researches, though of great importance in the history of the foundations of geometry, lacked the thoroughness which we would have expected had the author been a mathematician by profession.

The whole question was considered over again from a severely mathematical point of view by Sophus LIE[1] (1842-1899), who reduced the idea of motions to transformations between systems of coordinates, and congruence to invariance under such transformations. The underlying idea is that of a *group of transformations.*

Suppose we have a set of operations R, S, T, ... such that (1) the operation R followed by the operation S is again an operation (denoted by the product RS) of the set, and (2) $(RS)T = R(ST)$, then the set of operations is said to form a *group.* The operation, if it exists, which leaves the operand

[1] S. Lie, *Theorie der Transformationsgruppen*, vol. iii. (Leipzig, 1893), Abt. V. Kap. 20-24 ; and "Über die Grundlagen der Geometrie," *Leipziger Berichte*, **42** (1890).

unaltered, is called the *identical transformation*, and is denoted by 1.

Thus, if R, S, T are the operations of rotation about a fixed point through 1, 2 and 3 right angles, the operations 1, R, S, T form a group, and this is a *sub-group* of the group consisting of the 8 operations of rotation through every multiple of $\frac{\pi}{4}$.

The transformations which Lie considers are *infinitesimal transformations*, and the groups are *continuous groups*, such as the group of *all* the rotations about a fixed point. All the transformations which change points into points, straight lines into straight lines, and planes into planes form a continuous group which is called the general projective group.

The assumption from which Lie starts in his geometrical investigation is the "axiom of free mobility in the infinitesimal":

"If, at least within a certain region, a point P and a line-element through P are fixed, continuous motion is still possible, but if, in addition, a plane-element through P is fixed, no motion is possible."

Starting then with the group of projective transformations, he determines the character of the transformations so that this assumption may be verified, and he proves that they form a group which leaves unaltered either a non-ruled surface of the second degree (real or imaginary ellipsoid, hyperboloid of two sheets or elliptic paraboloid), or a plane and an imaginary conic lying on this plane. This invariant figure is just the *Absolute*. The motions of space, therefore, form a sub-group of the general projective group of point-transformations which leave the Absolute invariant. And so, without Helmholtz's axiom of monodromy, but using a definite assumption of free mobility, Lie establishes that the only possible types of metrical geometry are the three in which the absolute is a real non-ruled quadric (hyperbolic geometry), an imaginary quadric (elliptic geometry), and a plane with an imaginary conic (euclidean geometry).

6. Assumption of coordinates.

There are several points on which the investigations of Riemann, Helmholtz and Lie admit of criticism. The outstanding difficulty which strikes one at once lies in the use of coordinates. How can we define the coordinates of a point before we have fixed the idea of congruence? This question has been settled by an appeal to the famous procedure of von Staudt (1798-1867), the founder of projective geometry. He has shown[1] how, by means of repeated application of the quadrilateral-construction for a harmonic range (see Chap. III. § 5), numbers may be assigned to all the points of a line. This, and other questions involved, have now been solved by the modern procedure of Pasch, Hilbert and the Italian school represented by Pieri. This procedure, which marks a return to the classical method of Euclid, consists in developing geometry as a purely logical system deduced from an appropriately chosen system of axioms or assumptions.

7. Space-curvature and the fourth dimension.

A misunderstanding, which is especially common among philosophers, has grown around Riemann's use of the term "curvature." Helmholtz, whose philosophical

[1] G. K. Ch. v. Staudt, *Geometrie der Lage*, Nürnberg, 1847, and *Beiträge zur Geometrie der Lage*, Nürnberg, 1856-57-60.

writings[1] are much better known than his mathematical researches, has unfortunately contributed largely to this error. The use of the term "space-curvature" has led to the idea that non-euclidean geometry of three dimensions necessarily implies space of four dimensions, for curvature of space has no meaning except in relation to a fourth dimension. But when we assert that space has only three dimensions, we thereby deny that space has four dimensions. The geometry of this space of three dimensions, whether it is euclidean or non-euclidean, follows logically from certain assumed premises, one of which will certainly be equivalent to the statement that space has not more than three dimensions (cf. Chap. II. § 14, footnote). The origin of the fallacy lies in the failure to recognise that the geometry on a curved surface is nothing but a representation of the non-euclidean geometry.

This is brought out still more clearly by the fact that, as non-euclidean geometry, elliptic or hyperbolic, can be represented on certain curved surfaces in euclidean space, the converse is also true, that euclidean geometry can be represented on certain curved surfaces in elliptic or hyperbolic space; and, of course, we do not consider the euclidean plane as being a curved surface.

While, therefore, the conception of non-euclidean space of three dimensions in no way implies necessarily space-curvature or a fourth dimension, it is still an interesting speculation to suppose that we exist really in a space of four dimensions, but with our experience confined to a certain curved locus in this space, just as Helmholtz's "two-dimensional beings" were confined to the surface

[1] H. v. Helmholtz, "The origin and meaning of geometrical axioms," *Mind*, **1** (1876), **3** (1878); also in *Popular Scientific Lectures* (London, 1881), vol. ii.

of a sphere in space of three dimensions, and acquired in this way the idea that their geometry is non-euclidean.

W. K. Clifford [1] has gone further than this and imagined that the phenomena of electricity, etc., might be explained by periodic variations in the curvature of space. But we cannot now say that this three-dimensional universe in which we have our experience is *space* in the old sense, for space, as distinct from matter, consists of a changeless set of terms in changeless relations. There are two alternatives. We must either conceive that space is really of four dimensions and our universe is an extended sheet of matter existing in this space, the aether [2] if we like; and then, just as a plane surface is to our three-dimensional intelligence a pure abstraction, so our whole universe will become an ideal abstraction existing only in a mind that perceives space of four dimensions—an argument which has been brought to the support of Bishop Berkeley! [3] Or, we must resist our innate tendencies to separate out space and bodies as distinct entities, and attempt to build up a monistic theory of the physical world in terms of a single set of entities, material points, conceived as altering their relations with time.[4] In either case it is not space that is altering its qualities, but matter which is changing its form or relations with time.

[1] *The Common Sense of the Exact Sciences* (London, 1885), chap. iv. § 19.

[2] Cf. W. W. Rouse Ball, "A hypothesis relating to the nature of the ether and gravity," *Messenger of Math.*, **21** (1891).

[3] See C. H. Hinton, *Scientific Romances*, First Series, p. 31 (London, 1886). For other four-dimensional theories of physical phenomena see Hinton, *The Fourth Dimension* (London, 1904).

[4] Cf. A. N. Whitehead, "On mathematical concepts of the material world," *Phil. Trans.*, A **205** (1906).

8. Proof of the consistency of non-euclidean geometry.

The characteristic feature of the second period in the history of non-euclidean geometry is brought out for the first time by BELTRAMI[1] (1835-1900), who showed that Lobachevsky's geometry is represented upon a surface of constant curvature. This is historically the first euclidean representation of non-euclidean geometry, and is of importance in providing a proof of the consistency of the non-euclidean systems. While the development of hyperbolic geometry in the hands of Lobachevsky and Bolyai led to no apparent internal contradiction, a doubt remained that inconsistencies might yet be discovered if the investigations were pushed far enough. This doubt was removed by Beltrami's concrete representation by means of the pseudosphere, which reduced the consistency of non-euclidean geometry to depend upon that of euclidean geometry, which everyone admits to be self-consistent.

Any concrete representation of non-euclidean geometry in euclidean space can be applied with the same object. In fact, the Cayley representation is more suitable for this purpose, since it affords an equally good representation of three-dimensional geometry. The advantage of Beltrami's representation is that distances and angles are truly represented, and the arbitrariness which may perhaps be felt in the logarithmic expressions for distances and angles is eliminated.

At the present time no *absolute* test of consistency is

[1] E. Beltrami, *Saggio di interpretazione della geometria non-euclidea*, Naples, 1868. Beltrami also showed that, since the equation of a geodesic in geodesic coordinates is linear, the surface can be represented on a plane, geodesics being represented by straight lines, and real points being represented by points lying within a fixed circle. He thus gave the transition from the geodesic to the projective representation of Cayley.

known to exist, and the only test which we can apply is to construct a concrete representation by means of a body of propositions whose consistency is universally granted. In the case of non-euclidean geometry the test which has just been applied suffices to prove the impossibility of demonstrating Euclid's postulate. For, if Euclid's postulate could be mathematically or logically proved, this would establish an inconsistency in the non-euclidean systems; but any such inconsistency would appear again in the concrete representation. The mathematical truth of the euclidean and the non-euclidean geometries is equally strong.

9. Which is the true geometry?

There being no *a priori* means of deciding from the mathematical or logical side which of the three forms of geometry does in actual fact represent the true relations of things, three questions arise:

(1) Can the question of the true geometry be decided *a posteriori*, or experimentally?

(2) Can it be decided on philosophical grounds?

(3) Is it, after all, a proper question to ask, one to which an answer can be expected?

10. Attempts to determine the space-constant by astronomical measurements.

Let us consider what form of experiment we can contrive to determine, if possible, the geometrical character of space. Essentially it must consist in the measurements of distances and angles, the sort of triangulation which is employed to determine the figure of the earth, but on a prodigiously larger scale. If we could measure the angles