PART III.

CARDINAL ARITHMETIC.

SUMMARY OF PART III.

In this Part, we shall be concerned, first, with the definition and general logical properties of cardinal numbers (Section A); then with the operations of addition, multiplication and exponentiation, of which the definitions and formal laws do not require any restriction to finite numbers (Section B); then with the theory of finite and infinite, which is rendered somewhat complicated by the fact that there are two different senses of "finite," which cannot (so far as is known) be identified without assuming the multiplicative axiom. The theory of finite and infinite will be resumed, in connection with series, in Part V, Section E.

It is in this Part that the theory of types first becomes practically relevant. It will be found that contradictions concerning the maximum cardinal are solved by this theory. We have therefore devoted our first section in this Part (with the exception of two numbers giving the most elementary properties of cardinals in general, and of 0 and 1 and 2, respectively) to the application of types to cardinals. Every cardinal is typically ambiguous, and we confer typical definiteness by the notations of $*63$, $*64$, and $*65$. It is especially where existence-theorems are concerned that the theory of types is essential. The chief importance of the propositions of the present part lies, not only, as throughout the book, in the hypotheses necessary to secure the conclusions, but also in the typical ambiguity which can be allowed to the symbols consistently with the truth of the propositions in all the cases thereby included.

SECTION A.

DEFINITION AND LOGICAL PROPERTIES OF CARDINAL NUMBERS.

Summary of Section A.

The Cardinal Number of a class α, which we will denote by "$\mathrm{Nc}`\alpha$," is defined as the class of all classes similar to α, *i.e.* as $\hat{\beta}(\beta \,\mathrm{sm}\, \alpha)$. This definition is due to Frege, and was first published in his *Grundlagen der Arithmetik*[*]; its symbolic expression and use are to be found in his *Grundgesetze der Arithmetik*[†]. The chief merits of this definition are (1) that the formal properties which we expect cardinal numbers to have result from it; (2) that unless we adopt this definition or some more complicated and practically equivalent definition, it is necessary to regard the cardinal number of a class as an indefinable. Hence the above definition avoids a useless indefinable with its attendant primitive propositions.

It will be observed that, if x is any object, 1 is not the cardinal number of x, but that of $\iota`x$. This obviates a confusion which otherwise is liable to arise in dealing with classes. Suppose we have a class $\alpha$ consisting of many terms; we say, nevertheless, that it is *one* class. Thus it seems to be at once one and many. But in fact it is $\alpha$ that is many, and $\iota`\alpha$ that is one. In regard to zero, the analogous point is still clearer. Suppose we say "there are no Kings of France." This is equivalent to "the class of Kings of France has no members," or, in our language, "the class of Kings of France is a member of the class 0." It is obvious that we cannot say "the King of France is a member of the class 0," because there is no King of France. Thus in the case of 0 and 1, as more evidently in all other cases, a cardinal number appertains to a class, not to the members of the class.

For the purposes of formal definition, we subject the formula

$$\mathrm{Nc}`\alpha = \hat{\beta}(\beta \,\mathrm{sm}\, \alpha)$$

to some simplification. It will be seen that, according to this formula, "Nc" is a relation, namely the relation of a cardinal number to any class of which it is the number. Thus for example 1 has to $\iota`x$ the relation Nc; so has

* Breslau, 1884. Cf. especially pp. 79, 80.

† Jena, Vol. I. 1893, Vol. II. 1903. Cf. Vol. I. §§ 40—42, pp. 57, 58. The grounds in favour of this definition will be found at length in *Principles of Mathematics*, Part II.

2 to $\iota\text{‘}x \cup \iota\text{‘}y$, provided $x \neq y$. The relation Nc is, in fact, the relation $\overrightarrow{\mathrm{sm}}$; for $\overrightarrow{\mathrm{sm}}\text{‘}\alpha = \hat{\beta}\,(\beta \,\mathrm{sm}\, \alpha)$. Hence for formal purposes of definition we put

$$\mathrm{Nc} = \overrightarrow{\mathrm{sm}} \quad \mathrm{Df.}$$

The class of cardinal numbers is the class of objects which are the cardinal numbers of something or other, *i.e.* of objects which, for some α, are equal to $\mathrm{Nc}\text{‘}\alpha$. We call the class of cardinal numbers NC; thus we have

$$\mathrm{NC} = \hat{\mu}\,\{(\exists\alpha)\,.\,\mu = \mathrm{Nc}\text{‘}\alpha\}.$$

For purposes of formal definition, we replace this by the simpler formula

$$\mathrm{NC} = \mathrm{D}\text{‘}\mathrm{Nc} \quad \mathrm{Df.}$$

In the present section, we shall be concerned with what we may call the purely logical properties of cardinal numbers, namely those which do not depend upon the arithmetical operations of addition, multiplication and exponentiation, nor upon the distinction of finite and infinite*. The chief point to be dealt with, as regards both importance and difficulty, is the relation of a cardinal number in one type to the same or an associated cardinal number in another type. When a symbol is ambiguous as to type, we will call it *typically ambiguous*; when, either always or in a given context, it is unambiguous as to type, we will call it *typically definite*. Now the symbol "sm" is typically ambiguous; the only limitation on its type is that its domain and converse domain must both consist of classes. When we have $\alpha \,\mathrm{sm}\, \beta$, α and β need not be of the same type, in fact, in any type of classes, there are classes similar to some of the classes of any other type of classes. For example, we have $\iota\text{‘}x \,\mathrm{sm}\, \iota\text{‘}y$, whatever types x and y may belong to. This ambiguity of "sm" is derived from that of $1 \rightarrow 1$, which in turn is derived from that of 1. We denote (cf. ∗65·01) by "1_α" all the unit classes which are of the same type as α. Then (according to the definition ∗70·01) $1_\alpha \rightarrow 1_\beta$ will be the class of those one-one relations whose domain is of the same type as α and whose converse domain is of the same type as β. Thus "$1_\alpha \rightarrow 1_\beta$" is typically definite as soon as α and β are given. Suppose now, instead of having merely $\gamma \,\mathrm{sm}\, \delta$, we have

$$(\exists R)\,.\,R \in 1_\alpha \rightarrow 1_\beta\,.\,\mathrm{D}\text{‘}R = \gamma\,.\,\text{Ɑ}\text{‘}R = \delta\,;$$

then we know not only that $\gamma \,\mathrm{sm}\, \delta$, but also that γ belongs to the same type as α, and δ belongs to the same type as β. When the ambiguous symbol "sm" is rendered typically definite by having its domain defined as being of the same type as α, and its converse domain defined as being of the same type as β, we write it "$\mathrm{sm}_{(\alpha,\beta)}$," because generally, in accordance with ∗65·1, if R is a typically ambiguous relation, we write $R_{(\alpha,\beta)}$ for the typically

* The definitions of the arithmetical operations, and of finite and infinite, are really just as purely logical as what precedes them; but if we are to draw a line between logic and arithmetic somewhere, the arithmetical operations seem the natural point at which to place the beginning of arithmetic.

definite relation that results when the domain of R is to consist of terms of the same type as α, and the converse domain is to consist of terms of the same type as β. Thus we have

$$\gamma \operatorname{sm}_{(\alpha,\beta)} \delta \,.\equiv.\, (\exists R) \,.\, R \epsilon 1_\alpha \rightarrow 1_\beta \,.\, \gamma = \text{D}\text{‘}R \,.\, \delta = \text{Ɑ}\text{‘}R.$$

Here everything is typically definite if α and β (or their types) are given.

Passing now to the relation "Nc," it will be seen that it shares the typical ambiguity of "sm." In order to render it typically definite, we must derive it from a typically definite "sm." So long as nothing is added to give typical definiteness, "Nc‘γ" will mean all the classes belonging to some one (unspecified) type and similar to γ. If α is a member of the type to which these classes are to belong, then Nc‘γ is contained in the type of α. For this case, it is convenient to introduce the following two notations, already defined in ∗65. When a typically ambiguous relation R is to be rendered typically definite as to its domain only, by deciding that every member of the domain is to be *contained in* the type of α, we write "$R(\alpha)$" in place of R. When we further wish to determine R as having members of the converse domain *contained in* the type of β, we write "$R(\alpha, \beta)$" in place of R; and when we wish members of the converse domain to be *members of* the type of β, we write "$R(\alpha_\beta)$" in place of R. Thus

$$\text{sg}\text{‘}\{R_{(\alpha,\beta)}\} = \{\text{sg}\text{‘}R\}(\alpha_\beta)$$

(cf. ∗65·2), and in particular, since $\text{Nc} = \overrightarrow{\text{sm}}$,

$$\text{Nc}(\alpha_\beta) = \text{sg}\text{‘}\text{sm}_{(\alpha,\beta)}.$$

Thus "$\text{Nc}(\alpha_\beta)\text{‘}\gamma$" is only significant when γ is of the same type as β, and then it means "classes of the same type as α and similar to γ (which is of the same type as β)."

"$\text{Nc}(\alpha)\text{‘}\gamma$" will mean "classes of the same type as α and similar to γ." As soon as the types of α and γ are known, this is a typically definite symbol, being in fact equal to $\text{Nc}(\alpha_\gamma)\text{‘}\gamma$. Hence so long as we only wish to consider "Nc‘γ," typical definiteness is secured by writing "$\text{Nc}(\alpha)$" in place of "Nc."

When we come to the consideration of NC, "$\text{Nc}(\alpha)$" is no longer a sufficient determination, although it suffices to determine the type. Suppose we put

$$\text{NC}^\beta(\alpha) = \text{D}\text{‘}\text{Nc}(\alpha_\beta) \quad \text{Df};$$

we have also, in virtue of the definitions in ∗65,

$$\text{NC}(\alpha) = \text{NC} \cap t^2\text{‘}\alpha = \text{D}\text{‘}\text{Nc}(\alpha).$$

Thus $\text{NC}(\alpha)$ is definite as to type, but is the domain of a relation whose converse domain is ambiguous as to type; and it will appear that there are some propositions about $\text{NC}(\alpha)$ whose truth or falsehood depends upon the determination chosen for the converse domain of $\text{Nc}(\alpha)$. Hence if we wish to have a symbol which is completely definite, we must write "$\text{NC}^\beta(\alpha)$."

This point is important in connection with the contradictions as to the maximum cardinal. The following remarks will illustrate it further.

Cantor has shown that, if β is any class, no class contained in β is similar to $\mathrm{Cl}`\beta$. Hence in particular if $\beta$ is a type, no class contained in $\beta$ is similar to $\mathrm{Cl}`\beta$, which is the next type above β. Consequently, if $\beta = \alpha \cup -\alpha$, where α is any class, we have

$$\sim (\exists \gamma) \,.\, \gamma \subset \alpha \cup -\alpha \,.\, \gamma \operatorname{sm} \mathrm{Cl}`(\alpha \cup -\alpha).$$

Now (cf. $*63$) we put

$$t_0`\alpha = \alpha \cup -\alpha \quad \mathrm{Df},$$

and we have $t`\alpha = \mathrm{Cl}`(\alpha \cup -\alpha)$. Thus we find

$$\sim (\exists \gamma) \,.\, \gamma \subset t_0`\alpha \,.\, \gamma \operatorname{sm} t`\alpha.$$

Hence $$\mathrm{Nc}\,(\alpha_{t`\alpha})`t`\alpha = \Lambda.$$

That is to say, no class of the same type as α has as many members as $t`\alpha$ has. Hence also

$$\Lambda \in \mathrm{NC}^{t`\alpha}(\alpha).$$

But $$\gamma \subset t_0`\alpha \,.\, \supset \,.\, \gamma \in \mathrm{Nc}\,(\alpha_\alpha)`\gamma \,.\, \supset \,.\, \exists\,!\,\mathrm{Nc}\,(\alpha_\alpha)`\gamma,$$

and "$\mathrm{Nc}\,(\alpha_\alpha)`\gamma$" is only significant when $\gamma \subset t_0`\alpha$; hence

$$\mu \in \mathrm{NC}^\alpha(\alpha) \,.\, \supset_\mu \,.\, \exists\,!\,\mu$$

and $$\Lambda \sim\in \mathrm{NC}^\alpha(\alpha).$$

Now the notation "$\mathrm{NC}\,(\alpha)$" will apply with equal justice to $\mathrm{NC}^\alpha(\alpha)$ or to $\mathrm{NC}^{t`\alpha}(\alpha)$; but we have just seen that in the first case we shall have $\Lambda \sim\in \mathrm{NC}\,(\alpha)$, and in the second we shall have $\Lambda \in \mathrm{NC}\,(\alpha)$. Consequently "$\mathrm{NC}\,(\alpha)$" has not sufficient definiteness to prevent practically important differences between the various determinations of which it is capable.

A converse procedure to the above yields similar results. Let α be a class of classes; then $s`\alpha$ is of lower type than $\alpha$. Let us consider $\mathrm{NC}^{s`\alpha}(\alpha)$. In accordance with $*63$, we write $t_1`\alpha$ for the type containing $s`\alpha$, *i.e.* for $s`\alpha \cup -s`\alpha$. Then the greatest number in the class $\mathrm{NC}^{s`\alpha}(\alpha)$ will be $\mathrm{Nc}\,(\alpha)`t_1`\alpha$; but neither this nor any lesser member of the class will be equal to $\mathrm{Nc}\,(\alpha)`t_0`\alpha$, because, as before,

$$\sim (\exists \gamma) \,.\, \gamma \subset t_1`\alpha \,.\, \gamma \operatorname{sm} t_0`\alpha.$$

Hence $\mathrm{Nc}\,(\alpha)`t_0`\alpha$, which is a member of $\mathrm{NC}^\alpha(\alpha)$, is not a member of $\mathrm{NC}^{s`\alpha}(\alpha)$; but $\mathrm{NC}^\alpha(\alpha)$ and $\mathrm{NC}^{s`\alpha}(\alpha)$ have an equal right to be called $\mathrm{NC}\,(\alpha)$. Hence again "$\mathrm{NC}\,(\alpha)$" is a symbol not sufficiently definite for many of our purposes.

The solution of the paradox concerning the maximum cardinal is evident in view of what has been said. This paradox is as follows: It results from a theorem of Cantor's that there is no maximum cardinal, since, for all values of α,

$$\mathrm{Nc}`\mathrm{Cl}`\alpha > \mathrm{Nc}`\alpha.$$

But at first sight it would seem that the class which contains everything must be the greatest possible class, and must therefore contain the greatest possible number of terms. We have seen, however, that a class α must always be contained within some one type; hence all that is proved is that there are greater classes in the next type, which is that of $\mathrm{Cl}{\text{‘}}\alpha$. Since there is always a next higher type, we thus have a maximum cardinal in each type, without having any absolutely maximum cardinal. The maximum cardinal in the type of α is

$$\mathrm{Nc}\,(\alpha){\text{‘}}(\alpha \cup -\alpha).$$

But if we take the corresponding cardinal in the next type, *i.e.*

$$\mathrm{Nc}\,(\mathrm{Cl}{\text{‘}}\alpha){\text{‘}}(\alpha \cup -\alpha),$$

this is not as great as $\mathrm{Nc}\,(\mathrm{Cl}{\text{‘}}\alpha){\text{‘}}\mathrm{Cl}{\text{‘}}(\alpha \cup -\alpha)$, and is therefore not the maximum cardinal of its type. This gives the complete solution of the paradox.

For most purposes, what we wish to know in order to have a sufficient amount of typical definiteness is not the absolute types of α and β, as above, but merely what we may call their *relative* types. Thus, for example, α and β may be of the same type; in that case, $\mathrm{Nc}\,(\alpha_\beta)$ and $\mathrm{NC}^\beta\,(\alpha)$ are respectively equal to $\mathrm{Nc}\,(\alpha_\alpha)$ and $\mathrm{NC}^\alpha\,(\alpha)$. We will call cardinals which, for some α, are members of the class $\mathrm{NC}^\alpha\,(\alpha)$, *homogeneous* cardinals, because the "sm" from which they are derived is a homogeneous relation. We shall denote the homogeneous cardinal of α by "$\mathrm{N_0c}{\text{‘}}\alpha$," and we shall denote the class of homogeneous cardinals (in an unspecified type) by "$\mathrm{N_0C}$"; thus we put

$$\mathrm{N_0c}{\text{‘}}\alpha = \mathrm{Nc}{\text{‘}}\alpha \cap t{\text{‘}}\alpha \quad \mathrm{Df},$$

$$\mathrm{N_0C} = \mathrm{D}{\text{‘}}\mathrm{N_0c} \qquad \mathrm{Df}.$$

Almost all the properties of $\mathrm{N_0C}$ are the same in different types. When further typical definiteness is required, it can be secured by writing $\mathrm{N_0c}\,(\alpha)$, $\mathrm{N_0C}\,(\alpha)$ in place of $\mathrm{N_0c}$, $\mathrm{N_0C}$. For although $\mathrm{Nc}\,(\alpha)$ and $\mathrm{NC}\,(\alpha)$ were not wholly definite, $\mathrm{N_0c}\,(\alpha)$ and $\mathrm{N_0C}\,(\alpha)$ are wholly definite. Apart from the fact of being of different types, the only property in which $\mathrm{N_0C}\,(\alpha)$ and $\mathrm{N_0C}\,(\beta)$ differ when α and β are of different types is in regard to the magnitude of the cardinals belonging to them. Thus suppose the whole universe consisted (as monists aver) of a single individual. Let us call the type of this individual "Indiv." Then $\mathrm{N_0C}\,(\mathrm{Indiv})$ will consist of 0 and 1, *i.e.*

$$\mathrm{N_0C}\,(\mathrm{Indiv}) = \iota{\text{‘}}0 \cup \iota{\text{‘}}1.$$

But in the next higher type, there will be two members, namely Λ and Indiv. Thus

$$\mathrm{N_0C}\,(t{\text{‘}}\mathrm{Indiv}) = \iota{\text{‘}}0 \cup \iota{\text{‘}}1 \cup \iota{\text{‘}}2.$$

Similarly $\quad \mathrm{N_0C}\,(t{\text{‘}}t{\text{‘}}\mathrm{Indiv}) = \iota{\text{‘}}0 \cup \iota{\text{‘}}1 \cup \iota{\text{‘}}2 \cup \iota{\text{‘}}3 \cup \iota{\text{‘}}4,$

the members of $t'\,t'\mathrm{Indiv}$ being $\Lambda \cap t'\mathrm{Indiv}$, $\iota'\Lambda$, $\iota'\mathrm{Indiv}$, $\iota'\Lambda \cup \iota'\mathrm{Indiv}$; and so on. (The greatest cardinal in any except the lowest type is always a power of 2.)

The maximum of $N_0C(\alpha)$ is $N_0c'\,t_0'\alpha$; but apart from this difference of maximum and its consequences, $N_0C(\alpha)$ and $N_0C(\beta)$ do not differ in any important properties. Hence for most purposes N_0C and N_0c have as much typical definiteness as is necessary.

Among cardinals which are not homogeneous we shall consider three kinds. The first of these we shall call *ascending* cardinals. A cardinal $NC^{\beta}(\alpha)$ is called an *ascending* cardinal if the type of β is $t'\alpha$ or $t'\,t'\alpha$ or $t'\,t'\,t'\alpha$ or etc. We write $t^2{}'\alpha$ for $t'\,t'\alpha$, $t^3{}'\alpha$ for $t'\,t'\,t'\alpha$, and so on. We put

$$N^1c'\alpha = Nc'\alpha \cap t'\,t'\alpha \quad \mathrm{Df}$$

$$N^2c'\alpha = Nc'\alpha \cap t'\,t^2{}'\alpha \quad \mathrm{Df}$$

$$N^3c'\alpha = Nc'\alpha \cap t'\,t^3{}'\alpha \quad \mathrm{Df} \quad \text{and so on,}$$

and

$$N^1C = D'N^1c \quad \mathrm{Df}$$

$$N^2C = D'N^2c \quad \mathrm{Df}$$

$$N^3C = D'N^3c \quad \mathrm{Df} \quad \text{and so on.}$$

We then have obviously

$$N^1C(t'\alpha) \subset N_0C(t'\alpha).$$

We also have (by what was said earlier)

$$N_0c'\,t'\alpha \sim\epsilon\, N^1C(t'\alpha).$$

Hence

$$\exists !\, N_0C(t'\alpha) - N^1C(t'\alpha).$$

The members of $N_0C(t'\alpha) - N^1C(t'\alpha)$ will be all cardinals which exceed $Nc'\,t_0'\alpha$ but do not exceed $Nc'\,t'\alpha$.

Let us recur in illustration to our previous hypothesis of the universe consisting of a single individual. Then $N^1c'\mathrm{Indiv}$ will consist of those classes which are similar to "Indiv" but of the next higher type. These are $\iota'\Lambda$ and $\iota'\mathrm{Indiv}$. In our case we had $N_0c'\mathrm{Indiv} = 1$. This leads to

$$N^1c'\mathrm{Indiv} = 1 \,.\, N^2c'\mathrm{Indiv} = 1 \text{ etc.}$$

or, introducing typical definiteness,

$$N^1c'\mathrm{Indiv} = 1\,(t'\mathrm{Indiv}) \,.\, N^2c'\mathrm{Indiv} = 1\,(t^2{}'\mathrm{Indiv}) \text{ etc.}$$

We have then $1\,(t'\mathrm{Indiv}) \,\epsilon\, N^1C(t'\,t'\mathrm{Indiv})$. Also

$$1\,(t'\mathrm{Indiv}) \,\epsilon\, N_0C(t'\,t'\mathrm{Indiv}).$$

And in the case supposed, $1\,(t'\mathrm{Indiv})$ is the maximum of $N^1C(t'\,t'\mathrm{Indiv})$, but $2\,(t'\mathrm{Indiv}) \,\epsilon\, N_0C(t'\,t'\mathrm{Indiv})$. Hence

$$N_0C(t'\,t'\mathrm{Indiv}) - N^1C(t'\,t'\mathrm{Indiv}) = \iota'2.$$

Generalizing, we see that $N^1C(t'\alpha)$ consists of the same numbers as $N_0C(\alpha)$ each raised one degree in type. Similar propositions hold of $N^2C(t^2{}'\alpha)$, $N^3C(t^3{}'\alpha)$ etc.

It is often useful to have a notation for what we may call "the same cardinal in another type." Suppose μ is a typically definite cardinal; then we will denote by $\mu^{(1)}$ the same cardinal in the next type, *i.e.*

$$\mathrm{sm}``\mu \cap t`\mu.$$

Note that, if μ is a cardinal, $\mathrm{sm}``\mu \cap \mu = \mu$; and whether μ is a typically definite cardinal or not,

$$\mathrm{sm}``\mu \cap t`\alpha$$

is a cardinal in a definite type. If μ is typically definite, then $\mathrm{sm}``\mu \cap t`\alpha$ is wholly definite; if $\mu$ is typically ambiguous, $\mathrm{sm}``\mu \cap t`\alpha$ has the same kind of indefiniteness as belongs to $\mathrm{NC}(\alpha)$. The most important case is when μ is typically definite and α has an assigned relation of type to μ. We then put, as observed above,

$$\mu^{(1)} = \mathrm{sm}``\mu \cap t`\mu \quad \mathrm{Df}$$

$$\mu^{(2)} = \mathrm{sm}``\mu \cap t^{2}`\mu \quad \mathrm{Df\ etc.}$$

If μ is an $\mathrm{N_0C}$, $\mu^{(1)}$ is an $\mathrm{N^1C}$ and $\mu^{(2)}$ is an $\mathrm{N^2C}$ and so on. $\mathrm{N^1C}(t`\alpha)$ will consist of all numbers which are of the form $\mu^{(1)}$ for some μ which is a member of $\mathrm{N_0C}(\alpha)$; *i.e.*

$$\mathrm{N^1C}(t`\alpha) = \hat{\nu}\{(\exists\mu) \,.\, \mu \in \mathrm{N_0C}(\alpha) \,.\, \nu = \mu^{(1)}\}.$$

The second kind of non-homogeneous cardinals to be considered is called the class of "descending cardinals." These are such as go into a lower type; *i.e.* $\mathrm{Nc}(\alpha)`\beta$ is a descending cardinal if α is of a lower type than β. We put

$$\mathrm{N_1c}`\alpha = \mathrm{Nc}`\alpha \cap t`t_1`\alpha \quad \mathrm{Df}$$

$$\mathrm{N_2c}`\alpha = \mathrm{Nc}`\alpha \cap t`t_2`\alpha \quad \mathrm{Df\ etc.}$$

$$\mathrm{N_1C} = \mathrm{D}`\mathrm{N_1c} \quad \mathrm{Df}$$

$$\mathrm{N_2C} = \mathrm{D}`\mathrm{N_2c} \quad \mathrm{Df\ etc.}$$

$$\mu_{(1)} = \mathrm{sm}``\mu \cap t_1`\mu \quad \mathrm{Df}$$

$$\mu_{(2)} = \mathrm{sm}``\mu \cap t_2`\mu \quad \mathrm{Df\ etc.}$$

We have obviously $\qquad \mathrm{N_0c}`\alpha = \mathrm{N_1c}`\iota``\alpha.$

Hence $\qquad \mathrm{N_0C}(\alpha) \subset \mathrm{N_1C}(\alpha).$

Also $\qquad \gamma \in \mathrm{N_1c}`\delta \,.\supset .\, \mathrm{N_1c}`\delta = \mathrm{N_0c}`\gamma,$

whence $\qquad \exists ! \,\mathrm{N_1c}`\delta \,.\supset .\, \mathrm{N_1c}`\delta \in \mathrm{N_0C},$

whence $\qquad \mathrm{N_1C} - \iota`\Lambda \subset \mathrm{N_0C}.$

Since also $\Lambda \sim\in \mathrm{N_0C}(\alpha)$, we find

$$\mathrm{N_0C} = \mathrm{N_1C} - \iota`\Lambda,$$

this proposition not requiring any further typical definiteness, since it holds however such definiteness may be introduced, remembering that such definiteness is necessarily so introduced as to secure significance. Further, in virtue of the fact that no class contained in $t_0`\alpha$ is similar to $t`\alpha$, we have

$$\Lambda \in \mathrm{N_1C}(\alpha).$$

Consequently $\qquad \mathrm{N_1C} = \mathrm{N_0C} \cup \iota\text{‘}\Lambda.$

We can prove in just the same way

$$\mathrm{N_2C} = \mathrm{N_0C} \cup \iota\text{‘}\Lambda.$$

Hence $\qquad \mathrm{N_1C} = \mathrm{N_2C},$

and this result can obviously be extended to all descending cardinals.

The third kind of non-homogeneous cardinals to be considered may be called "relational cardinals." They are those applicable to classes of relations having a given relation of type to a given class. Consider for example $\mathrm{Nc}\text{‘}\epsilon_\Delta\text{‘}\kappa$. (We shall take this as the definition of the product of the numbers of the members of κ.) Suppose now that κ consists of a single term: we want to be able to say

$$\mathrm{Nc}\text{‘}\epsilon_\Delta\text{‘}\kappa = \mathrm{Nc}\text{‘}\breve{\iota}\text{‘}\kappa.$$

We have in this case, if $\kappa = \iota\text{‘}\alpha$,

$$\epsilon_\Delta\text{‘}\kappa = \downarrow \alpha\text{‘‘}\alpha,$$

and we know that $\downarrow \alpha\text{‘‘}\alpha \,\mathrm{sm}\, \alpha$. But if we put simply

$$\mathrm{Nc}\text{‘} \downarrow \alpha\text{‘‘}\alpha = \mathrm{Nc}\text{‘}\alpha,$$

our proposition, though not mistaken, requires care in interpretation. Just as we put $\iota\text{‘‘}\alpha \in \mathrm{N^1c}\text{‘}\alpha$, so we want a notation giving typical definiteness to the proposition $\downarrow \alpha\text{‘‘}\alpha \in \mathrm{Nc}\text{‘}\alpha$. This is provided as follows.

Using the notation of $*64$, put

$$\begin{array}{ll} \mathrm{N_{00}c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}t_{00}\text{‘}\alpha & \mathrm{Df} \\ \mathrm{N_0{}^1c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha \cap t\text{‘}t_0{}^1\text{‘}\alpha & \mathrm{Df\ etc.} \\ \mathrm{N_{00}C} = \mathrm{D}\text{‘}\mathrm{N_{00}c} & \mathrm{Df} \\ \mathrm{N_0{}^1C} = \mathrm{D}\text{‘}\mathrm{N_0{}^1c} & \mathrm{Df\ etc.} \\ \mu_{(00)} = \mathrm{sm}\text{‘‘}\mu \cap t\text{‘}t_{00}\text{‘}t_1\text{‘}\mu & \mathrm{Df\ etc.} \end{array}$$

Then we have, for example,

$$\downarrow \alpha\text{‘‘}\alpha \subset t_0{}^1\text{‘}\alpha, \ \textit{i.e.}\ \downarrow \alpha\text{‘‘}\alpha \in t\text{‘}t_0{}^1\text{‘}\alpha.$$

Hence $\downarrow \alpha\text{‘‘}\alpha \in \mathrm{N_0{}^1c}\text{‘}\alpha$, where $\mathrm{N_0{}^1c}\text{‘}\alpha = \mathrm{Nc}\text{‘}\alpha \cap t_0{}^1\text{‘}\alpha$.

Similarly $\qquad x \in t\text{‘}\alpha \,.\supset.\, \downarrow x\text{‘‘}\alpha \in \mathrm{N_{00}c}\text{‘}\alpha.$

Thus the above definitions give us what is required.

In order to complete our notation for types, we should need to be able to express the type of the domain or converse domain of R, or of any relation whose domain and converse domain have respectively given relations of type to the domain and converse domain of R. Thus we might put

$$\begin{array}{ll} \mathrm{d_0}\text{‘}R = t_0\text{‘}\mathrm{D}\text{‘}R & \mathrm{Df} \\ \mathrm{b_0}\text{‘}R = t_0\text{‘}\text{Ɑ}\text{‘}R & \mathrm{Df} \end{array}$$

("b" appears here as "d" written backwards)

$$\begin{array}{ll} \mathrm{d_{00}}\text{‘}R = t\text{‘}(\mathrm{d_0}\text{‘}R \uparrow \mathrm{b_0}\text{‘}R) & \mathrm{Df} \\ \qquad = t\text{‘}R & \\ \mathrm{d}^{mn}\text{‘}R = t\text{‘}(t^m\text{‘}\mathrm{d_0}\text{‘}R \uparrow t^n\text{‘}\mathrm{b_0}\text{‘}R) & \mathrm{Df\ and\ so\ on.} \end{array}$$

This notation would enable us to deal with descending relational cardinals. But it is not required in the present work, and is therefore not introduced among the numbered propositions.

When a typically ambiguous symbol, such as "sm" or "Nc," occurs more than once in a given context, it must not be assumed, unless required by the conditions of significance, that it is to receive the same typical determination in each case. Thus *e.g.* we shall write "$\alpha \operatorname{sm} \beta . \supset . \beta \operatorname{sm} \alpha$," although, if α and β are of different types, the two symbols "sm" must receive different typical determinations.

Formulae which are typically ambiguous, or only partially definite as to type, must not be admitted unless every significant interpretation is true. Thus for example we may admit

$$\text{"} \vdash . \alpha \in \mathrm{Nc}\text{‘}\alpha \text{"}$$

because here "Nc" must mean "$\mathrm{Nc}(\alpha_\alpha)$," so that the only ambiguity remaining is as to the type of α, and the formula holds whatever type α may belong to, provided "$\mathrm{Nc}\text{‘}\alpha$" is significant, *i.e.* provided α is a class. But we must not, from "$\alpha \in \mathrm{Nc}\text{‘}\alpha$," allow ourselves to infer

$$\text{"} \exists ! \mathrm{Nc}\text{‘}\alpha. \text{"}$$

For here the conditions of significance no longer demand that "Nc" should mean "$\mathrm{Nc}(\alpha_\alpha)$": it might just as well mean "$\mathrm{Nc}(\beta_\alpha)$." And as we saw, if β is a lower type than α, and α is sufficiently large of its type, we may have

$$\mathrm{Nc}(\beta_\alpha)\text{‘}\alpha = \Lambda,$$

so that "$\exists ! \mathrm{Nc}\text{‘}\alpha$" is not admissible without qualification. Nevertheless, as we shall see in *100, there are a certain number of propositions to be made about a wholly ambiguous Nc or NC.

$*100$. DEFINITION AND ELEMENTARY PROPERTIES OF CARDINAL NUMBERS.

Summary of $*100$.

In this number we shall be concerned only with such immediate consequences of the definition of cardinal numbers as do not require typical definiteness, beyond what the inherent conditions of significance may bestow. We introduce here the fundamental definitions:

$*100{\cdot}01$. $\mathrm{Nc} = \overrightarrow{\mathrm{sm}}$ Df

$*100{\cdot}02$. $\mathrm{NC} = \mathrm{D}\text{‘}\mathrm{Nc}$ Df

The definition "Nc" is required chiefly for the sake of the descriptive function $\mathrm{Nc}\text{‘}\alpha$. We have

$*100{\cdot}1$. $\vdash . \mathrm{Nc}\text{‘}\alpha = \hat{\beta}\,(\beta \,\mathrm{sm}\, \alpha) = \hat{\beta}\,(\alpha \,\mathrm{sm}\, \beta)$

This may be stated in various equivalent forms, which are given at the beginning of this number ($*100{\cdot}1$—$\cdot16$). After a few propositions on Nc as a relation, we proceed to the elementary properties of $\mathrm{Nc}\text{‘}\alpha$. We have

$*100{\cdot}3$. $\vdash . \alpha \in \mathrm{Nc}\text{‘}\alpha$

$*100{\cdot}31$. $\vdash : \alpha \in \mathrm{Nc}\text{‘}\beta . \equiv . \beta \in \mathrm{Nc}\text{‘}\alpha . \equiv . \alpha \,\mathrm{sm}\, \beta$

$*100{\cdot}321$. $\vdash : \alpha \,\mathrm{sm}\, \beta . \supset . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta$

$*100{\cdot}33$. $\vdash : \exists ! \,\mathrm{Nc}\text{‘}\alpha \cap \mathrm{Nc}\text{‘}\beta . \supset . \alpha \,\mathrm{sm}\, \beta$

We proceed next to the elementary properties of NC. We have

$*100{\cdot}4$. $\vdash : \mu \in \mathrm{NC} . \equiv . (\exists \alpha) . \mu = \mathrm{Nc}\text{‘}\alpha$

$*100{\cdot}42$. $\vdash : \mu, \nu \in \mathrm{NC} . \exists ! \,\mu \cap \nu . \supset . \mu = \nu$

$*100{\cdot}45$. $\vdash : \mu \in \mathrm{NC} . \alpha \in \mu . \supset . \mathrm{Nc}\text{‘}\alpha = \mu$

$*100{\cdot}51$. $\vdash : \mu \in \mathrm{NC} . \alpha \in \mu . \supset . \mathrm{sm}\text{“}\mu = \mathrm{Nc}\text{‘}\alpha$

Observe that when we have such a hypothesis as "$\mu \in \mathrm{NC}$," the μ, though it may be of any type, must be of *some* type; hence the μ cannot have the typical ambiguity which belongs to $\mathrm{Nc}\text{‘}\alpha$. If we put $\mu = \mathrm{Nc}\text{‘}\alpha$, this will hold only in the type of μ; but "$\mathrm{sm}\text{“}\mu$" is a typically ambiguous symbol, which will represent in any type the "same" number as μ. Thus "$\text{sm}``\mu = \text{Nc}`\alpha$" is an equation which is applicable to all possible typical determinations of "sm" and "Nc."

$*100{\cdot}52.\quad \vdash : \mu \epsilon \text{NC} . \exists ! \mu . \supset . \text{sm}``\mu \epsilon \text{NC}$

The hypothesis $\exists ! \mu$ is unnecessary, but we cannot prove this till later ($*102$).

We end the number with some propositions ($*100{\cdot}6$—$\cdot64$) stating that various classes (such as $\iota``\alpha$), which have already been proved to be similar to α, have $\text{Nc}`\alpha$ members.

$*100{\cdot}01.\quad \text{Nc} = \overrightarrow{\text{sm}} \quad \text{Df}$

$*100{\cdot}02.\quad \text{NC} = \text{D}`\text{Nc} \quad \text{Df}$

$*100{\cdot}1.\quad \vdash . \text{Nc}`\alpha = \hat{\beta}(\beta \,\text{sm}\, \alpha) = \hat{\beta}(\alpha \,\text{sm}\, \beta) \quad [*32{\cdot}13 . *73{\cdot}31 . (*100{\cdot}01)]$

$*100{\cdot}11.\quad \vdash . \text{Nc}`\alpha = \hat{\beta}\{(\exists R) . R \epsilon 1 \rightarrow 1 . \text{D}`R = \alpha . \text{G}`R = \beta\} \quad [*100{\cdot}1 . *73{\cdot}1]$

$*100{\cdot}12.\quad \vdash . \text{Nc}`\alpha = \hat{\beta}\{(\exists R) . R \epsilon 1 \rightarrow 1 . \alpha \subset \text{D}`R . \beta = \breve{R}``\alpha\}$
$[*100{\cdot}1 . *73{\cdot}11]$

$*100{\cdot}13.\quad \vdash . \text{Nc}`\alpha = \text{G}``(1 \rightarrow 1 \cap \overleftarrow{\text{D}}`\alpha) = \text{D}``(1 \rightarrow 1 \cap \overleftarrow{\text{G}}`\alpha)$

Dem.

$\vdash . *100{\cdot}11 . *33{\cdot}6 . \quad \supset \vdash . \text{Nc}`\alpha = \hat{\beta}\{(\exists R) . R \epsilon 1 \rightarrow 1 . R \epsilon \overleftarrow{\text{D}}`\alpha . \text{G}`R = \beta\}$

$[*22{\cdot}33 . *37{\cdot}6] \quad = \text{G}``(1 \rightarrow 1 \cap \overleftarrow{\text{D}}`\alpha) \quad (1)$

$\vdash . *100{\cdot}1 . *73{\cdot}1 . *33{\cdot}61 . \supset \vdash . \text{Nc}`\alpha = \hat{\beta}\{(\exists R) . R \epsilon 1 \rightarrow 1 . R \epsilon \overleftarrow{\text{G}}`\alpha . \text{D}`R = \beta\}$

$[*22{\cdot}33 . *37{\cdot}6] \quad = \text{D}``(1 \rightarrow 1 \cap \overleftarrow{\text{G}}`\alpha) \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*100{\cdot}14.\quad \vdash . \text{Nc}`\alpha = \hat{\beta}\{(\exists R) . \alpha \subset \text{G}`R . R \upharpoonright \alpha \epsilon 1 \rightarrow 1 . \beta = R``\alpha\}$
$[*73{\cdot}15 . *100{\cdot}1]$

$*100{\cdot}15.\quad \vdash . \text{Nc}`\alpha = \hat{\beta}\{(\exists R) : \text{E}!! R``\alpha :$
$x, y \epsilon \alpha . R`x = R`y . \supset_{x,y} . x = y : \beta = R``\alpha\}$

Dem.

$\vdash . *74{\cdot}1{\cdot}11 . \supset$

$\vdash :. \text{E}!! R``\alpha : x, y \epsilon \alpha . R`x = R`y . \supset_{x,y} . x = y : \beta = R``\alpha : \equiv :$
$R \upharpoonright \alpha \epsilon 1 \rightarrow \text{Cls} . \alpha \subset \text{G}`R . R \upharpoonright \alpha \epsilon 1 \rightarrow 1 . \beta = R``\alpha \quad (1)$

$\vdash . (1) . *4{\cdot}71 . *100{\cdot}14 . \supset \vdash . \text{Prop}$

$*100{\cdot}16.\quad \vdash . \text{Nc}`\alpha = \hat{\beta}\{(\exists R) :. x, y \epsilon \alpha . \supset_{x,y} : R`x = R`y . \equiv . x = y :. \beta = R``\alpha\}$

Dem.

$\vdash . *71{\cdot}59 . \supset$

$\vdash :: x, y \epsilon \alpha . \supset_{x,y} : R`x = R`y . \equiv . x = y :. \equiv . R \upharpoonright \alpha \epsilon 1 \rightarrow 1 . \alpha \subset \text{G}`R \quad (1)$

$\vdash . (1) . *100{\cdot}14 . \supset \vdash . \text{Prop}$

***100·2.** $\vdash . \mathrm{E} ! \mathrm{Nc}`\alpha$ [*32·12 . (*100·01)]

***100·21.** $\vdash . \mathrm{G}`\mathrm{Nc} = \mathrm{Cls}$

Dem.

$$\vdash . *37{\cdot}76 . (*100{\cdot}01) . \supset \vdash . \mathrm{G}`\mathrm{Nc} \subset \mathrm{Cls} \quad (1)$$
$$\vdash . *33{\cdot}431 . *100{\cdot}2 . \supset \vdash . \mathrm{Cls} \subset \mathrm{G}`\mathrm{Nc} \quad (2)$$
$$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$$

***100·22.** $\vdash . \mathrm{Nc} \in 1 \rightarrow \mathrm{Cls}$ [*72·12 . (*100·01)]

***100·3.** $\vdash . \alpha \in \mathrm{Nc}`\alpha$ [*73·3 . *100·1]

Note that it is fallacious to infer $\exists ! \mathrm{Nc}`\alpha$, for reasons explained in the introduction to the present section.

***100·31.** $\vdash : \alpha \in \mathrm{Nc}`\beta . \equiv . \beta \in \mathrm{Nc}`\alpha . \equiv . \alpha \,\mathrm{sm}\, \beta$ [*32·18 . *73·31 . (*100·01)]

***100·32.** $\vdash : \alpha \in \mathrm{Nc}`\beta . \beta \in \mathrm{Nc}`\gamma . \supset . \alpha \in \mathrm{Nc}`\gamma$ [*100·31 . *73·32]

***100·321.** $\vdash : \alpha \,\mathrm{sm}\, \beta . \supset . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta$

Dem.

$$\vdash . *73{\cdot}37 . \supset \vdash :. \mathrm{Hp} . \supset : \gamma \,\mathrm{sm}\, \alpha . \equiv_\gamma . \gamma \,\mathrm{sm}\, \beta :$$
$$[*100{\cdot}1] \qquad \supset : \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta :. \supset \vdash . \mathrm{Prop}$$

Note that $\mathrm{Nc}`\alpha = \mathrm{Nc}`\beta . \supset . \alpha \,\mathrm{sm}\, \beta$ is not always true. We might be tempted to prove it as follows:

$$\vdash . *100{\cdot}1 . \supset \vdash :. \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta . \equiv : \gamma \,\mathrm{sm}\, \alpha . \equiv_\gamma . \gamma \,\mathrm{sm}\, \beta :$$
$$[*10{\cdot}1] \qquad \supset : \alpha \,\mathrm{sm}\, \alpha . \equiv . \alpha \,\mathrm{sm}\, \beta :$$
$$[*73{\cdot}3] \qquad \supset : \alpha \,\mathrm{sm}\, \beta$$

But the use of *10·1 here is only legitimate when the "sm" concerned is a homogeneous relation. If $\mathrm{Nc}`\alpha$, $\mathrm{Nc}`\beta$ are descending cardinals, we may have $\mathrm{Nc}`\alpha = \Lambda = \mathrm{Nc}`\beta$ without having $\alpha \,\mathrm{sm}\, \beta$.

***100·33.** $\vdash : \exists ! \mathrm{Nc}`\alpha \cap \mathrm{Nc}`\beta . \supset . \alpha \,\mathrm{sm}\, \beta$

Dem.

$$\vdash . *100{\cdot}1 . \supset \vdash : \mathrm{Hp} . \supset . (\exists \gamma) . \gamma \,\mathrm{sm}\, \alpha . \gamma \,\mathrm{sm}\, \beta .$$
$$[*73{\cdot}31] \qquad \supset . (\exists \gamma) . \alpha \,\mathrm{sm}\, \gamma . \gamma \,\mathrm{sm}\, \beta .$$
$$[*73{\cdot}32] \qquad \supset . \alpha \,\mathrm{sm}\, \beta : \supset \vdash . \mathrm{Prop}$$

Note that we do not always have

$$\alpha \,\mathrm{sm}\, \beta . \supset . \exists ! \mathrm{Nc}`\alpha \cap \mathrm{Nc}`\beta.$$

For if the Nc concerned is a descending Nc, and α and β are sufficiently great, $\mathrm{Nc}`\alpha$ and $\mathrm{Nc}`\beta$ may both be Λ. For example, we have

$$\mathrm{Cl}`(\alpha \cup -\alpha) \,\mathrm{sm}\, \mathrm{Cl}`(\alpha \cup -\alpha).$$

But $\mathrm{Nc}\,(\alpha)`\mathrm{Cl}`(\alpha \cup -\alpha) = \Lambda$, so that

$$\sim \exists ! \mathrm{Nc}\,(\alpha)`\mathrm{Cl}`(\alpha \cup -\alpha) \cap \mathrm{Nc}\,(\alpha)`\mathrm{Cl}`(\alpha \cup -\alpha).$$

Thus "$\alpha \operatorname{sm} \beta . \supset . \exists ! \mathrm{Nc}`\alpha \cap \mathrm{Nc}`\beta$" is not always true when it is significant.

***100·34.** $\vdash : \exists ! \mathrm{Nc}`\alpha \cap \mathrm{Nc}`\beta . \supset . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta$ [*100·33·321]

***100·35.** $\vdash :. \exists ! \mathrm{Nc}`\alpha . \vee . \exists ! \mathrm{Nc}`\beta : \supset :$
$\mathrm{Nc}`\alpha = \mathrm{Nc}`\beta . \equiv . \alpha \in \mathrm{Nc}`\beta . \equiv . \beta \in \mathrm{Nc}`\alpha . \equiv . \alpha \operatorname{sm} \beta$

Dem.

$\vdash . *22·5 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta . \supset . \exists ! \mathrm{Nc}`\alpha \cap \mathrm{Nc}`\beta .$
[*100·33] $\quad \supset . \alpha \operatorname{sm} \beta$ (1)
$\vdash . (1) . *100·321 . \supset \vdash :. \mathrm{Hp} . \supset : \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta . \equiv . \alpha \operatorname{sm} \beta$ (2)
$\vdash . (2) . *100·31 . \supset \vdash . \mathrm{Prop}$

Thus the only case in which the implications in *100·321·33·34 cannot be turned into equivalences is the case in which $\mathrm{Nc}`\alpha$ and $\mathrm{Nc}`\beta$ are both Λ.

***100·36.** $\vdash :. \beta \in \mathrm{Nc}`\alpha . \supset : \exists ! \alpha . \equiv . \exists ! \beta$ [*100·31 . *73·36]

***100·4.** $\vdash : \mu \in \mathrm{NC} . \equiv . (\exists \alpha) . \mu = \mathrm{Nc}`\alpha$ [*37·78·79 . (*100·02·01)]

***100·41.** $\vdash . \mathrm{Nc}`\alpha \in \mathrm{NC}$ [*100·4·2 . *14·204]

***100·42.** $\vdash : \mu, \nu \in \mathrm{NC} . \exists ! \mu \cap \nu . \supset . \mu = \nu$

Dem.

$\vdash . *100·4 . \supset \vdash : \mathrm{Hp} . \supset . (\exists \alpha, \beta) . \mu = \mathrm{Nc}`\alpha . \nu = \mathrm{Nc}`\beta . \exists ! \mathrm{Nc}`\alpha \cap \mathrm{Nc}`\beta .$
[*100·34] $\quad \supset . (\exists \alpha, \beta) . \mu = \mathrm{Nc}`\alpha . \nu = \mathrm{Nc}`\beta . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta .$
[*14·15] $\quad \supset . \mu = \nu : \supset \vdash . \mathrm{Prop}$

***100·43.** $\vdash . \mathrm{NC} \in \mathrm{Cls}^2 \operatorname{excl}$ [*100·42 . *84·11]

***100·44.** $\vdash :. \mu \in \mathrm{NC} . \exists ! \mathrm{Nc}`\alpha . \supset : \alpha \in \mu . \equiv . \mathrm{Nc}`\alpha = \mu$

Dem.

$\vdash . *100·3 . \supset \vdash : \mathrm{Nc}`\alpha = \mu . \supset . \alpha \in \mu$ (1)
$\vdash . *10·24 . \supset \vdash : \mu \in \mathrm{NC} . \exists ! \mathrm{Nc}`\alpha . \alpha \in \mu . \supset .$
$\mu \in \mathrm{NC} . \exists ! \mu . \exists ! \mathrm{Nc}`\alpha . \alpha \in \mu .$
[*100·4] $\quad \supset . (\exists \beta) . \mu = \mathrm{Nc}`\beta . \exists ! \mathrm{Nc}`\beta . \exists ! \mathrm{Nc}`\alpha . \alpha \in \mathrm{Nc}`\beta .$
[*100·35] $\quad \supset . (\exists \beta) . \mu = \mathrm{Nc}`\beta . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta .$
[*14·15] $\quad \supset . \mathrm{Nc}`\alpha = \mu$ (2)
$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

***100·45.** $\vdash : \mu \in \mathrm{NC} . \alpha \in \mu . \supset . \mathrm{Nc}`\alpha = \mu$ [*100·4·31·321]

***100·5.** $\vdash : \mu \in \mathrm{NC} . \alpha, \beta \in \mu . \supset . \alpha \operatorname{sm} \beta$

Dem.

$\vdash . *100·4 . \supset \vdash : \mathrm{Hp} . \supset . (\exists \gamma) . \mu = \mathrm{Nc}`\gamma . \alpha, \beta \in \mathrm{Nc}`\gamma .$
[*100·31] $\quad \supset . (\exists \gamma) . \alpha \operatorname{sm} \gamma . \beta \operatorname{sm} \gamma .$
[*73·31·32] $\quad \supset . \alpha \operatorname{sm} \beta : \supset \vdash . \mathrm{Prop}$

$*100\cdot51$. $\vdash : \mu \in \mathrm{NC} . \alpha \in \mu . \supset . \mathrm{sm}``\mu = \mathrm{Nc}`\alpha$

Dem.

$\vdash . *100\cdot5 . \mathrm{Fact} . \supset \vdash :. \mathrm{Hp} . \supset : \beta \in \mu . \gamma \,\mathrm{sm}\, \beta . \supset . \alpha \,\mathrm{sm}\, \beta . \gamma \,\mathrm{sm}\, \beta .$

$[*73\cdot31\cdot32] \quad \supset . \alpha \,\mathrm{sm}\, \gamma .$

$[*100\cdot31] \quad \supset . \gamma \in \mathrm{Nc}`\alpha \quad (1)$

$\vdash . (1) . *10\cdot11\cdot21\cdot23 . *37\cdot1 . \supset \vdash : \mathrm{Hp} . \supset . \mathrm{sm}``\mu \subset \mathrm{Nc}`\alpha \quad (2)$

$\vdash . *100\cdot31 . \quad \supset \vdash :. \mathrm{Hp} . \supset : \gamma \in \mathrm{Nc}`\alpha . \supset . \gamma \,\mathrm{sm}\, \alpha . \alpha \in \mu .$

$[*37\cdot1] \quad \supset . \gamma \in \mathrm{sm}``\mu \quad (3)$

$\vdash . (2) . (3) . \supset \vdash . \mathrm{Prop}$

$*100\cdot511$. $\vdash : \exists ! \mathrm{Nc}`\beta . \supset . \mathrm{sm}``\mathrm{Nc}`\beta = \mathrm{Nc}`\beta$

Here the last "$\mathrm{Nc}`\beta$" may be of a different type from the others: the proposition holds however its type is determined.

Dem.

$\vdash . *100\cdot51\cdot41 . \supset \vdash : \alpha \in \mathrm{Nc}`\beta . \supset . \mathrm{sm}``\mathrm{Nc}`\beta = \mathrm{Nc}`\alpha$

$[*100\cdot31\cdot321] \quad = \mathrm{Nc}`\beta \quad (1)$

$\vdash . (1) . *10\cdot11\cdot23 . \supset \vdash . \mathrm{Prop}$

$*100\cdot52$. $\vdash : \mu \in \mathrm{NC} . \exists ! \mu . \supset . \mathrm{sm}``\mu \in \mathrm{NC} \quad [*100\cdot51\cdot4]$

This proposition still holds when $\mu = \Lambda$, but the proof is more difficult, since it depends upon the proof that every null-class of classes is an NC, which in turn depends upon the proof that $\mathrm{Cl}`\alpha$ is not similar to α or to any class contained in α.

$*100\cdot521$. $\vdash : \mu \in \mathrm{NC} . \exists ! \mathrm{sm}``\mu . \supset . \mathrm{sm}``\mathrm{sm}``\mu = \mu$

Dem.

$\vdash . *37\cdot29 . \mathrm{Transp} . \supset \vdash :. \mathrm{Hp} . \supset : \exists ! \mu :$

$[*100\cdot52] \quad \supset : \mathrm{sm}``\mu \in \mathrm{NC} :$

$[*100\cdot51 . \mathrm{Hp}] \quad \supset : \gamma \in \mathrm{sm}``\mu . \supset . \mathrm{sm}``\mathrm{sm}``\mu = \mathrm{Nc}`\gamma \quad (1)$

$\vdash . *37\cdot1 . \mathrm{Fact} . \quad \supset \vdash : \mathrm{Hp} . \gamma \in \mathrm{sm}``\mu . \supset . (\exists \alpha) . \alpha \in \mu . \mu \in \mathrm{NC} . \gamma \,\mathrm{sm}\, \alpha .$

$[*100\cdot45\cdot321] \quad \supset . (\exists \alpha) . \mathrm{Nc}`\alpha = \mu . \mathrm{Nc}`\gamma = \mathrm{Nc}`\alpha .$

$[*13\cdot17] \quad \supset . \mathrm{Nc}`\gamma = \mu \quad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash :. \mathrm{Hp} . \gamma \in \mathrm{sm}``\mu . \supset . \mathrm{sm}``\mathrm{sm}``\mu = \mu \quad (3)$

$\vdash . (3) . *10\cdot11\cdot23\cdot35 . \supset \vdash . \mathrm{Prop}$

$*100\cdot53$. $\vdash :. \exists ! \mu . \exists ! \nu . \supset : \mu \in \mathrm{NC} . \nu = \mathrm{sm}``\mu . \equiv . \nu \in \mathrm{NC} . \mu = \mathrm{sm}``\nu$

Dem.

$\vdash . *100\cdot52 . \supset \vdash :. \mathrm{Hp} . \supset : \mu \in \mathrm{NC} . \nu = \mathrm{sm}``\mu . \supset . \nu \in \mathrm{NC} \quad (1)$

$\vdash . *100\cdot521 . \supset \vdash :. \mathrm{Hp} . \supset : \mu \in \mathrm{NC} . \nu = \mathrm{sm}``\mu . \supset . \mu = \mathrm{sm}``\nu \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :. \mathrm{Hp} . \supset : \mu \in \mathrm{NC} . \nu = \mathrm{sm}``\mu . \supset . \nu \in \mathrm{NC} . \mu = \mathrm{sm}``\nu \quad (3)$

$\vdash . (3) . (3) \dfrac{\nu, \mu}{\mu, \nu} . \supset \vdash . \mathrm{Prop}$

***100·6.** $\vdash . \iota``\alpha \in \mathrm{Nc}`\alpha$ [*73·41 . *100·31]

***100·61.** $\vdash . \hat{\beta}\{(\exists y) . y \in \alpha . \beta = \iota`x \cup \iota`y\} \in \mathrm{Nc}`\alpha$ [*73·27 . *54·21 . *100·31]

***100·62.** $\vdash . x \downarrow ``\alpha \in \mathrm{Nc}`\alpha$ [*73·61 . *100·31]

***100·621.** $\vdash . \downarrow x``\alpha \in \mathrm{Nc}`\alpha$ [*73·611 . *100·31]

***100·63.** $\vdash . \epsilon_\Delta`\iota`\alpha \in \mathrm{Nc}`\alpha$ [*83·41 . *100·31]

***100·631.** $\vdash . \mathrm{D}``\epsilon_\Delta`\iota`\alpha \in \mathrm{Nc}`\alpha$ [*83·7 . *100·6]

***100·64.** $\vdash : \kappa \in \mathrm{Cls}^2\,\mathrm{excl} . \supset . \mathrm{D}``\epsilon_\Delta`\kappa \subset \mathrm{Nc}`\kappa$

Dem.

$\vdash . *84{\cdot}3 . *80{\cdot}14 . \supset \vdash : \mathrm{Hp} . R \in \epsilon_\Delta`\kappa . \supset . R \in 1 \to 1 . \kappa = \mathrm{G}`R .$

[*73·2.*100·31] $\supset . \mathrm{D}`R \in \mathrm{Nc}`\kappa : \supset \vdash . \mathrm{Prop}$

$*101$. ON 0 AND 1 AND 2.

Summary of $*101$.

In the present number, we have to show that 0 and 1 and 2 as previously defined are cardinal numbers in the sense defined in $*100$, and to add a few elementary propositions to those already given concerning them. We prove ($*101{\cdot}12{\cdot}241$) that 0 and 1 are not null, which cannot be proved, with our axioms, for any other cardinal, except (in the case of finite cardinals) when the type is specified as a sufficiently high one. Thus we prove ($*101{\cdot}42{\cdot}43$) that 2_{Cls} and 2_{Rel} exist; this follows from $\Lambda \neq \mathrm{V}$ and $\dot{\Lambda} \neq \dot{\mathrm{V}}$. We prove ($*101{\cdot}22{\cdot}34$) that 0 and 1 and 2 are all different from each other. We prove ($*101{\cdot}15{\cdot}28$) that $\mathrm{sm}\text{``}0 = 0$ and $\mathrm{sm}\text{``}1 = 1$, but we cannot prove $\mathrm{sm}\text{``}2 = 2$ unless we assume the existence of at least two individuals, or define the first 2 in "$\mathrm{sm}\text{``}2 = 2$" as a 2 of some type other than 2_{Indiv}, where "Indiv" stands for the type of individuals.

It should be observed that, since 0 and 1 and 2 are typically ambiguous, their properties are analogous to those of "$\mathrm{Nc}\text{‘}\alpha$" rather than to those of μ, where $\mu \in \mathrm{NC}$. For example, we have

$*100{\cdot}511$. $\vdash : \exists ! \, \mathrm{Nc}\text{‘}\beta \,.\, \supset .\, \mathrm{sm}\text{``}\mathrm{Nc}\text{‘}\beta = \mathrm{Nc}\text{‘}\beta$

but we shall not have $\mu \in \mathrm{NC} \,.\, \exists ! \, \mu \,.\, \supset .\, \mathrm{sm}\text{``}\mu = \mu$ unless the "sm" concerned is homogeneous, since in other cases the symbols do not express a significant proposition. But in $*100{\cdot}511$ we may substitute 0 or 1 or 2, and the proposition remains significant and true. In fact we have ($*101{\cdot}1{\cdot}2{\cdot}31$)

$$\vdash .\, 0 = \mathrm{Nc}\text{‘}\Lambda \,.\, 1 = \mathrm{Nc}\text{‘}\iota\text{‘}x \,.\, 2 = \mathrm{Nc}\text{‘}(\iota\text{‘}\iota\text{‘}x \cup \iota\text{‘}\Lambda),$$

where 0 and 1 and 2 have an ambiguity corresponding to that of "Nc."

$*101{\cdot}1$. $\vdash .\, 0 = \mathrm{Nc}\text{‘}\Lambda$ [$*73{\cdot}48 \,.\, *100{\cdot}1$]

$*101{\cdot}11$. $\vdash .\, 0 \in \mathrm{NC}$ [$*101{\cdot}1 \,.\, *100{\cdot}4$]

$*101{\cdot}12$. $\vdash .\, \exists ! \, 0$ [$*51{\cdot}161 \,.\, (*54{\cdot}01)$]

$*101{\cdot}13$. $\vdash .\, \exists ! \, 0 \cap \mathrm{Cl}\text{‘}\alpha \,.\, \Lambda \in 0 \cap \mathrm{Cl}\text{‘}\alpha$ [$*51{\cdot}16 \,.\, *60{\cdot}3$]

***101·14.** $\vdash : \mathrm{Nc}`\gamma = 0 \,.\equiv .\, \gamma = \Lambda$

Dem.

$$\begin{array}{ll}
\vdash . *101{\cdot}1{\cdot}12 . \supset \vdash : \mathrm{Nc}`\gamma = 0 . \equiv . & \mathrm{Nc}`\gamma = \mathrm{Nc}`\Lambda \,.\, \exists ! \,\mathrm{Nc}`\Lambda \,. \\
[*13{\cdot}194] & \equiv . \mathrm{Nc}`\gamma = \mathrm{Nc}`\Lambda \,.\, \exists ! \,\mathrm{Nc}`\Lambda \,.\, \exists ! \,\mathrm{Nc}`\gamma \,. \\
[*100{\cdot}35] & \equiv . \gamma \in \mathrm{Nc}`\Lambda \,.\, \exists ! \,\mathrm{Nc}`\Lambda \,.\, \exists ! \,\mathrm{Nc}`\gamma \,. \\
[*101{\cdot}1 . *54{\cdot}102] & \equiv . \gamma = \Lambda \,.\, \exists ! \,\mathrm{Nc}`\Lambda \,.\, \exists ! \,\mathrm{Nc}`\gamma \,. \\
[*101{\cdot}1{\cdot}12 . *13{\cdot}194] & \equiv . \gamma = \Lambda : \supset \vdash . \mathrm{Prop}
\end{array}$$

***101·15.** $\vdash . \mathrm{sm}``0 = 0$

Dem.

$$\begin{array}{ll}
\vdash . *37{\cdot}1 . \supset \vdash : \gamma \in \mathrm{sm}``0 . \equiv . & (\exists \alpha) . \alpha \in 0 . \gamma \,\mathrm{sm}\, \alpha \,. \\
[*54{\cdot}102] & \equiv . \gamma \,\mathrm{sm}\, \Lambda \,. \\
[*73{\cdot}48] & \equiv . \gamma \in 0 : \supset \vdash . \mathrm{Prop}
\end{array}$$

***101·16.** $\vdash :. \mu \in \mathrm{NC} - \iota`0 . \supset : \alpha \in \mu . \supset_\alpha . \exists ! \,\alpha$

Dem.

$$\begin{array}{lll}
\vdash . *100{\cdot}45 . & \supset \vdash : \mu \in \mathrm{NC} . \Lambda \in \mu . \supset . \mu = \mathrm{Nc}`\Lambda & \\
[*101{\cdot}1] & \qquad\qquad\qquad\qquad = 0 & (1) \\
\vdash . (1) . \mathrm{Transp} . & \supset \vdash :. \mu \in \mathrm{NC} - \iota`0 . \supset : \Lambda \sim \in \mu : & \\
[*24{\cdot}63] & \qquad \supset : \alpha \in \mu . \supset_\alpha . \exists ! \,\alpha :. \supset \vdash . \mathrm{Prop} &
\end{array}$$

***101·17.** $\vdash : \Lambda \in \mathrm{Nc}`\alpha \,.\equiv .\, \mathrm{Nc}`\alpha = 0 \,.\equiv .\, \mathrm{Nc}`\alpha = \mathrm{Nc}`\Lambda \,.\equiv .\, \alpha = \Lambda$

Dem.

$$\begin{array}{lll}
\vdash . *100{\cdot}31{\cdot}321 . \supset \vdash : \Lambda \in \mathrm{Nc}`\alpha . & \supset . \mathrm{Nc}`\alpha = \mathrm{Nc}`\Lambda \,. & \\
[*101{\cdot}1] & \supset . \mathrm{Nc}`\alpha = 0 & (1) \\
\vdash . *101{\cdot}13 . \quad \supset \vdash : \mathrm{Nc}`\alpha = 0 . & \supset . \Lambda \in \mathrm{Nc}`\alpha & (2) \\
\vdash . (1) . (2) . \quad \supset \vdash : \Lambda \in \mathrm{Nc}`\alpha . & \equiv . \mathrm{Nc}`\alpha = 0 \,. & (3) \\
[*101{\cdot}1] & \equiv . \mathrm{Nc}`\alpha = \mathrm{Nc}`\Lambda \,. & (4) \\
[*101{\cdot}14] & \equiv . \alpha = \Lambda & (5) \\
\vdash . (3) . (4) . (5) . \supset \vdash . \mathrm{Prop} & &
\end{array}$$

***101·2.** $\vdash . 1 = \mathrm{Nc}`\iota`x$ $[*73{\cdot}45 . *100{\cdot}1]$

***101·21.** $\vdash . 1 \in \mathrm{NC}$ $[*101{\cdot}2 . *100{\cdot}4]$

***101·22.** $\vdash . 1 \neq 0$

Dem.

$$\begin{array}{ll}
\vdash . *52{\cdot}21 . *101{\cdot}13 . & \supset \vdash . \Lambda \sim \in 1 . \Lambda \in 0 \,. \\
[*13{\cdot}14] & \supset \vdash . 1 \neq 0
\end{array}$$

***101·23.** $\vdash . 1 \cap 0 = \Lambda$

Dem.

$$\begin{array}{lll}
\vdash . *52{\cdot}21 . & \supset \vdash : \alpha \in 1 . \supset . \alpha \neq \Lambda \,. & \\
[*54{\cdot}102] & \qquad\quad \supset . \alpha \sim \in 0 & (1) \\
\vdash . (1) . *24{\cdot}39 . \supset \vdash . \mathrm{Prop} & &
\end{array}$$