we can only legitimately assert "*any* value" if *all* values are true; for otherwise, since the value of the variable remains to be determined, it might be so determined as to give a false proposition. Thus in the above instance, since we have

$$\vdash . x = x$$

we may infer $\vdash . (x) . x = x.$

And generally, given an assertion containing a real variable $x$, we may transform the real variable into an apparent one by placing the $x$ in brackets at the beginning, followed by as many dots as there are after the assertion-sign.

When we assert something containing a real variable, we cannot strictly be said to be asserting a *proposition*, for we only obtain a definite proposition by assigning a value to the variable, and then our assertion only applies to one definite case, so that it has not at all the same force as before. When what we assert contains a real variable, we are asserting a wholly undetermined one of all the propositions that result from giving various values to the variable. It will be convenient to speak of such assertions as *asserting a propositional function.* The ordinary formulae of mathematics contain such assertions; for example

$$\text{"}\sin^2 x + \cos^2 x = 1\text{"}$$

does not assert this or that particular case of the formula, nor does it assert that the formula holds for *all* possible values of $x$, though it is equivalent to this latter assertion; it simply asserts that the formula holds, leaving $x$ wholly undetermined; and it is able to do this legitimately, because, however $x$ may be determined, a true proposition results.

Although an assertion containing a real variable does not, in strictness, assert a proposition, yet it will be spoken of as asserting a proposition except when the nature of the ambiguous assertion involved is under discussion.

*Definition and real variables.* When the *definiens* contains one or more real variables, the *definiendum* must also contain them. For in this case we have a function of the real variables, and the *definiendum* must have the same meaning as the *definiens* for all values of these variables, which requires that the symbol which is the *definiendum* should contain the letters representing the real variables. This rule is not always observed by mathematicians, and its infringement has sometimes caused important confusions of thought, notably in geometry and the philosophy of space.

In the definitions given above of "$p . q$" and "$p \supset q$" and "$p \equiv q$," $p$ and $q$ are real variables, and therefore appear on both sides of the definition. In the definition of "$\sim \{(x) . \phi x\}$" only the function considered, namely $\phi \hat{z}$, is a real variable; thus so far as concerns the rule in question, $x$ need not appear on the left. But when a real variable is a function, it is necessary to indicate how the argument is to be supplied, and therefore there are objections to omitting an apparent variable where (as in the case before us) this is the argument to the function which is the real variable. This appears more plainly if, instead of a general function $\phi\hat{x}$, we take some particular function, say "$\hat{x} = a$," and consider the definition of $\sim\{(x) \,.\, x = a\}$. Our definition gives

$$\sim\{(x) \,.\, x = a\} \,.=.\, (\exists x) \,.\, \sim(x = a) \quad \text{Df.}$$

But if we had adopted a notation in which the ambiguous value "$x = a$," containing the apparent variable $x$, did not occur in the *definiendum*, we should have had to construct a notation employing the function itself, namely "$\hat{x} = a$." This does not involve an apparent variable, but would be clumsy in practice. In fact we have found it convenient and possible—except in the explanatory portions—to keep the explicit use of symbols of the type "$\phi\hat{x}$," either as constants [*e.g.* $\hat{x} = a$] or as real variables, almost entirely out of this work.

*Propositions connecting real and apparent variables.* The most important propositions connecting real and apparent variables are the following:

(1) "When a propositional function can be asserted, so can the proposition that all values of the function are true." More briefly, if less exactly, "what holds of any, however chosen, holds of all." This translates itself into the rule that when a real variable occurs in an assertion, we may turn it into an apparent variable by putting the letter representing it in brackets immediately after the assertion-sign.

(2) "What holds of all, holds of any," *i.e.*

$$\vdash : (x) \,.\, \phi x \,.\supset.\, \phi y.$$

This states "if $\phi x$ is always true, then $\phi y$ is true."

(3) "If $\phi y$ is true, then $\phi x$ is sometimes true," *i.e.*

$$\vdash : \phi y \,.\supset.\, (\exists x) \,.\, \phi x.$$

An asserted proposition of the form "$(\exists x) \,.\, \phi x$" expresses an "existence-theorem," namely "there exists an $x$ for which $\phi x$ is true." The above proposition gives what is in practice the only way of proving existence-theorems: we always have to find some particular $y$ for which $\phi y$ holds, and thence to infer "$(\exists x) \,.\, \phi x$." If we were to assume what is called the multiplicative axiom, or the equivalent axiom enunciated by Zermelo, that would, in an important class of cases, give an existence-theorem where no particular instance of its truth can be found.

In virtue of "$\vdash : (x) \,.\, \phi x \,.\supset.\, \phi y$" and "$\vdash : \phi y \,.\supset.\, (\exists x) \,.\, \phi x$," we have "$\vdash : (x) \,.\, \phi x \,.\supset.\, (\exists x) \,.\, \phi x$," *i.e.* "what is always true is sometimes true." This would not be the case if nothing existed; thus our assumptions contain the assumption that there is something. This is involved in the principle that what holds of all, holds of any; for this would not be true if there were no "any."

(4) "If $\phi x$ is always true, and $\psi x$ is always true, then '$\phi x . \psi x$' is always true," *i.e.*

$$\vdash :. (x) . \phi x : (x) . \psi x : \supset . (x) . \phi x . \psi x.$$

(This requires that $\phi$ and $\psi$ should be functions which take arguments of the same *type*. We shall explain this requirement at a later stage.) The converse also holds; *i.e.* we have

$$\vdash :. (x) . \phi x . \psi x . \supset : (x) . \phi x : (x) . \psi x.$$

It is to some extent optional which of the propositions connecting real and apparent variables are taken as primitive propositions. The primitive propositions assumed, on this subject, in the body of the work (∗9), are the following:

(1) $$\vdash : \phi x . \supset . (\exists z) . \phi z.$$

(2) $$\vdash : \phi x \vee \phi y . \supset . (\exists z) . \phi z,$$

*i.e.* if either $\phi x$ is true, or $\phi y$ is true, then $(\exists z) . \phi z$ is true. (On the necessity for this primitive proposition, see remarks on ∗9·11 in the body of the work.)

(3) If we can assert $\phi y$, where $y$ is a real variable, then we can assert $(x) . \phi x$; *i.e.* what holds of any, however chosen, holds of all.

*Formal implication and formal equivalence.* When an implication, say $\phi x . \supset . \psi x$, is said to hold always, *i.e.* when $(x) : \phi x . \supset . \psi x$, we shall say that $\phi x$ *formally implies* $\psi x$; and propositions of the form "$(x) : \phi x . \supset . \psi x$" will be said to state *formal implications*. In the usual instances of implication, such as "'Socrates is a man' implies 'Socrates is mortal,'" we have a proposition of the form "$\phi x . \supset . \psi x$" in a case in which "$(x) : \phi x . \supset . \psi x$" is true. In such a case, we feel the implication as a particular case of a formal implication. Thus it has come about that implications which are not particular cases of formal implications have not been regarded as implications at all. There is also a practical ground for the neglect of such implications, for, speaking generally, they can only be *known* when it is already known either that their hypothesis is false or that their conclusion is true; and in neither of these cases do they serve to make us know the conclusion, since in the first case the conclusion need not be true, and in the second it is known already. Thus such implications do not serve the purpose for which implications are chiefly useful, namely that of making us know, by deduction, conclusions of which we were previously ignorant. *Formal* implications, on the contrary, do serve this purpose, owing to the psychological fact that we often know "$(x) : \phi x . \supset . \psi x$" and $\phi y$, in cases where $\psi y$ (which follows from these premisses) cannot easily be known directly.

These reasons, though they do not warrant the complete neglect of implications that are not instances of formal implications, are reasons which make formal implication very important. A formal implication states that, for all possible values of $x$, if the hypothesis $\phi x$ is true, the conclusion $\psi x$ is true. Since "$\phi x . \supset . \psi x$" will always be true when $\phi x$ is false, it is only the values of $x$ that make $\phi x$ true that are *important* in a formal implication; what is effectively stated is that, for all these values, $\psi x$ is true. Thus propositions of the form "all $\alpha$ is $\beta$," "no $\alpha$ is $\beta$" state formal implications, since the first (as appears by what has just been said) states

$$(x) : x \text{ is an } \alpha . \supset . x \text{ is a } \beta,$$

while the second states

$$(x) : x \text{ is an } \alpha . \supset . x \text{ is not a } \beta.$$

And any formal implication "$(x) : \phi x . \supset . \psi x$" may be interpreted as: "All values of $x$ which satisfy* $\phi x$ satisfy $\psi x$," while the formal implication "$(x) : \phi x . \supset . \sim\psi x$" may be interpreted as: "No values of $x$ which satisfy $\phi x$ satisfy $\psi x$."

We have similarly for "some $\alpha$ is $\beta$" the formula

$$(\exists x) . x \text{ is an } \alpha . x \text{ is a } \beta,$$

and for "some $\alpha$ is not $\beta$" the formula

$$(\exists x) . x \text{ is an } \alpha . x \text{ is not a } \beta.$$

Two functions $\phi x$, $\psi x$ are called *formally equivalent* when each always implies the other, *i.e.* when

$$(x) : \phi x . \equiv . \psi x,$$

and a proposition of this form is called a *formal equivalence*. In virtue of what was said about truth-values, if $\phi x$ and $\psi x$ are formally equivalent, either may replace the other in any truth-function. Hence for all the purposes of mathematics or of the present work, $\phi \hat{z}$ may replace $\psi \hat{z}$ or vice versa in any proposition with which we shall be concerned. Now to say that $\phi x$ and $\psi x$ are formally equivalent is the same thing as to say that $\phi \hat{z}$ and $\psi \hat{z}$ have the same *extension*, *i.e.* that any value of $x$ which satisfies either satisfies the other. Thus whenever a constant function occurs in our work, the truth-value of the proposition in which it occurs depends only upon the extension of the function. A proposition containing a function $\phi \hat{z}$ and having this property (*i.e.* that its truth-value depends only upon the extension of $\phi \hat{z}$) will be called an *extensional* function of $\phi \hat{z}$. Thus the functions of functions with which we shall be specially concerned will all be extensional functions of functions.

What has just been said explains the connection (noted above) between the fact that the functions of propositions with which mathematics is specially

* A value of $x$ is said to *satisfy* $\phi x$ or $\phi \hat{x}$ when $\phi x$ is true for that value of $x$.

concerned are all truth-functions and the fact that mathematics is concerned with extensions rather than intensions.

*Convenient abbreviation.* The following definitions give alternative and often more convenient notations:

$$\phi x \,.\, \supset_x .\, \psi x :=: (x) : \phi x \,.\, \supset .\, \psi x \quad \text{Df},$$

$$\phi x \,.\, \equiv_x .\, \psi x :=: (x) : \phi x \,.\, \equiv .\, \psi x \quad \text{Df}.$$

This notation "$\phi x \,.\, \supset_x .\, \psi x$" is due to Peano, who, however, has no notation for the general idea "$(x) \,.\, \phi x$." It may be noticed as an exercise in the use of dots as brackets that we might have written

$$\phi x \supset_x \psi x \,.=.\, (x) \,.\, \phi x \supset \psi x \quad \text{Df},$$

$$\phi x \equiv_x \psi x \,.=.\, (x) \,.\, \phi x \equiv \psi x \quad \text{Df}.$$

In practice however, when $\phi \hat{x}$ and $\psi \hat{x}$ are special functions, it is not possible to employ fewer dots than in the first form, and often more are required.

The following definitions give abbreviated notations for functions of two or more variables:

$$(x, y) \,.\, \phi(x, y) \,.=: (x) : (y) \,.\, \phi(x, y) \quad \text{Df},$$

and so on for any number of variables;

$$\phi(x, y) \,.\, \supset_{x,y} .\, \psi(x, y) :=: (x, y) : \phi(x, y) \,.\, \supset .\, \psi(x, y) \quad \text{Df},$$

and so on for any number of variables.

*Identity.* The propositional function "$x$ is identical with $y$" is expressed by

$$x = y.$$

This will be defined (cf. $*13{\cdot}01$), but, owing to certain difficult points involved in the definition, we shall here omit it (cf. Chapter II). We have, of course,

$$\vdash .\, x = x \quad \text{(the law of identity)},$$

$$\vdash : x = y \,.\, \equiv .\, y = x,$$

$$\vdash : x = y \,.\, y = z \,.\, \supset .\, x = z.$$

The first of these expresses the *reflexive* property of identity: a relation is called *reflexive* when it holds between a term and itself, either universally, or whenever it holds between that term and some term. The second of the above propositions expresses that identity is a *symmetrical* relation: a relation is called *symmetrical* if, whenever it holds between $x$ and $y$, it also holds between $y$ and $x$. The third proposition expresses that identity is a *transitive* relation: a relation is called *transitive* if, whenever it holds between $x$ and $y$ and between $y$ and $z$, it holds also between $x$ and $z$.

We shall find that no new definition of the sign of equality is required in mathematics: all mathematical equations in which the sign of equality is used in the ordinary way express some identity, and thus use the sign of equality in the above sense.

If $x$ and $y$ are identical, either can replace the other in any proposition without altering the truth-value of the proposition; thus we have

$$\vdash : x = y . \supset . \phi x \equiv \phi y.$$

This is a fundamental property of identity, from which the remaining properties mostly follow.

It might be thought that identity would not have much importance, since it can only hold between $x$ and $y$ if $x$ and $y$ are different symbols for the same object. This view, however, does not apply to what we shall call "descriptive phrases," *i.e.* "the so-and-so." It is in regard to such phrases that identity is important, as we shall shortly explain. A proposition such as "Scott was the author of Waverley" expresses an identity in which there is a descriptive phrase (namely "the author of Waverley"); this illustrates how, in such cases, the assertion of identity may be important. It is essentially the same case when the newspapers say "the identity of the criminal has not transpired." In such a case, the criminal is known by a descriptive phrase, namely "the man who did the deed," and we wish to find an $x$ of whom it is true that "$x =$ the man who did the deed." When such an $x$ has been found, the identity of the criminal has transpired.

*Classes and relations.* A *class* (which is the same as a *manifold* or *aggregate*) is all the objects satisfying some propositional function. If $\alpha$ is the class composed of the objects satisfying $\phi\hat{x}$, we shall say that $\alpha$ is the class *determined* by $\phi\hat{x}$. Every propositional function thus determines a class, though if the propositional function is one which is always false, the class will be *null, i.e.* will have no members. The class determined by the function $\phi\hat{x}$ will be represented by $\hat{z}\,(\phi z)$*. Thus for example if $\phi x$ is an equation, $\hat{z}\,(\phi z)$ will be the class of its roots; if $\phi x$ is "$x$ has two legs and no feathers," $\hat{z}\,(\phi z)$ will be the class of men; if $\phi x$ is "$0 < x < 1$," $\hat{z}\,(\phi z)$ will be the class of proper fractions, and so on.

It is obvious that the same class of objects will have many determining functions. When it is not necessary to specify a determining function of a class, the class may be conveniently represented by a single Greek letter. Thus Greek letters, other than those to which some constant meaning is assigned, will be exclusively used for classes.

There are two kinds of difficulties which arise in formal logic; one kind arises in connection with classes and relations and the other in connection with descriptive functions. The point of the difficulty for classes and relations, so far as it concerns classes, is that a class cannot be an object suitable as an argument to any of its determining functions. If $\alpha$ represents

* Any other letter may be used instead of $z$.

a class and $\phi\hat{x}$ one of its determining functions [so that $\alpha = \hat{z}(\phi z)$], it is not sufficient that $\phi\alpha$ be a false proposition, it must be nonsense. Thus a certain classification of what appear to be objects into things of essentially different types seems to be rendered necessary. This whole question is discussed in Chapter II, on the theory of types, and the formal treatment in the systematic exposition, which forms the main body of this work, is guided by this discussion. The part of the systematic exposition which is specially concerned with the theory of classes is $*20$, and in this Introduction it is discussed in Chapter III. It is sufficient to note here that, in the complete treatment of $*20$, we have avoided the decision as to whether a class of things has in any sense an existence as one object. A decision of this question in either way is indifferent to our logic, though perhaps, if we had regarded some solution which held classes and relations to be in some real sense objects as both true and likely to be universally received, we might have simplified one or two definitions and a few preliminary propositions. Our symbols, such as "$\hat{x}(\phi x)$" and $\alpha$ and others, which represent classes and relations, are merely defined in their use, just as $\nabla^2$, standing for

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

has no meaning apart from a suitable function of $x$, $y$, $z$ on which to operate. The result of our definitions is that the way in which we use classes corresponds in general to their use in ordinary thought and speech; and whatever may be the ultimate interpretation of the one is also the interpretation of the other. Thus in fact our classification of types in Chapter II really performs the single, though essential, service of justifying us in refraining from entering on trains of reasoning which lead to contradictory conclusions. The justification is that what seem to be propositions are really nonsense.

The definitions which occur in the theory of classes, by which the idea of a class (at least in use) is based on the other ideas assumed as primitive, cannot be understood without a fuller discussion than can be given now (cf. Chapter II of this Introduction and also $*20$). Accordingly, in this preliminary survey, we proceed to state the more important simple propositions which result from those definitions, leaving the reader to employ in his mind the ordinary unanalysed idea of a class of things. Our symbols in their usage conform to the ordinary usage of this idea in language. It is to be noticed that in the systematic exposition our treatment of classes and relations requires no new primitive ideas and only two new primitive propositions, namely the two forms of the "Axiom of Reducibility" (cf. next Chapter) for one and two variables respectively.

The propositional function "$x$ is a member of the class $\alpha$" will be expressed, following Peano, by the notation

$$x \,\epsilon\, \alpha.$$

Here $\epsilon$ is chosen as the initial of the word ἐστί. "$x \epsilon \alpha$" may be read "$x$ is an $\alpha$." Thus "$x \epsilon$ man" will mean "$x$ is a man," and so on. For typographical convenience we shall put

$$x \sim\epsilon\, \alpha \,.=.\, \sim(x \epsilon \alpha) \quad \text{Df},$$

$$x, y \,\epsilon\, \alpha \,.=.\, x \epsilon \alpha \,.\, y \epsilon \alpha \quad \text{Df}.$$

For "class" we shall write "Cls"; thus "$\alpha \epsilon$ Cls" means "$\alpha$ is a class."

We have

$$\vdash : x \epsilon \hat{z}(\phi z) \,.\equiv.\, \phi x,$$

*i.e.* "'$x$ is a member of the class determined by $\phi\hat{z}$' is equivalent to '$x$ satisfies $\phi\hat{z}$,' or to '$\phi x$ is true.'"

A class is wholly determinate when its membership is known, that is, there cannot be two different classes having the same membership. Thus if $\phi x$, $\psi x$ are formally equivalent functions, they determine the same class; for in that case, if $x$ is a member of the class determined by $\phi\hat{x}$, and therefore satisfies $\phi x$, it also satisfies $\psi x$, and is therefore a member of the class determined by $\psi\hat{x}$. Thus we have

$$\vdash :. \hat{z}(\phi z) = \hat{z}(\psi z) \,.\equiv :\, \phi x \,.\equiv_x .\, \psi x.$$

The following propositions are obvious and important:

$$\vdash :. \alpha = \hat{z}(\phi z) \,.\equiv :\, x \epsilon \alpha \,.\equiv_x .\, \phi x,$$

*i.e.* $\alpha$ is identical with the class determined by $\phi\hat{z}$ when, and only when, "$x$ is an $\alpha$" is formally equivalent to $\phi x$;

$$\vdash :. \alpha = \beta \,.\equiv :\, x \epsilon \alpha \,.\equiv_x .\, x \epsilon \beta,$$

*i.e.* two classes $\alpha$ and $\beta$ are identical when, and only when, they have the same membership;

$$\vdash .\, \hat{x}(x \epsilon \alpha) = \alpha,$$

*i.e.* the class whose determining function is "$x$ is an $\alpha$" is $\alpha$, in other words, $\alpha$ is the class of objects which are members of $\alpha$;

$$\vdash .\, \hat{z}(\phi z) \,\epsilon\, \text{Cls},$$

*i.e.* the class determined by the function $\phi\hat{z}$ is a class.

It will be seen that, according to the above, any function of one variable can be replaced by an equivalent function of the form "$x \epsilon \alpha$." Hence any extensional function of functions which holds when its argument is a function of the form "$\hat{z} \epsilon \alpha$," whatever possible value $\alpha$ may have, will hold also when its argument is any function $\phi\hat{z}$. Thus variation of classes can replace variation of functions of one variable in all the propositions of the sort with which we are concerned.

In an exactly analogous manner we introduce dual or dyadic relations, *i.e.* relations between two terms. Such relations will be called simply "relations"; relations between more than two terms will be distinguished as *multiple* relations, or (when the number of their terms is specified) as triple, quadruple,... relations, or as triadic, tetradic,... relations. Such relations will not concern us until we come to Geometry. For the present, the only relations we are concerned with are *dual* relations.

Relations, like classes, are to be taken in *extension*, *i.e.* if $R$ and $S$ are relations which hold between the same pairs of terms, $R$ and $S$ are to be identical. We may regard a relation, in the sense in which it is required for our purposes, as a class of couples; *i.e.* the couple $(x, y)$ is to be one of the class of couples constituting the relation $R$ if $x$ has the relation $R$ to $y$*. This view of relations as classes of couples will not, however, be introduced into our symbolic treatment, and is only mentioned in order to show that it is possible so to understand the meaning of the word *relation* that a relation shall be determined by its extension.

Any function $\phi(x, y)$ determines a relation $R$ between $x$ and $y$. If we regard a relation as a class of couples, the relation determined by $\phi(x, y)$ is the class of couples $(x, y)$ for which $\phi(x, y)$ is true. The relation determined by the function $\phi(x, y)$ will be denoted by

$$\hat{x}\hat{y}\phi(x, y).$$

We shall use a capital letter for a relation when it is not necessary to specify the determining function. Thus whenever a capital letter occurs, it is to be understood that it stands for a relation.

The propositional function "$x$ has the relation $R$ to $y$" will be expressed by the notation

$$xRy.$$

This notation is designed to keep as near as possible to common language, which, when it has to express a relation, generally mentions it between its terms, as in "$x$ loves $y$," "$x$ equals $y$," "$x$ is greater than $y$," and so on. For "relation" we shall write "Rel"; thus "$R \epsilon \mathrm{Rel}$" means "$R$ is a relation."

Owing to our taking relations in extension, we shall have

$$\vdash :. \hat{x}\hat{y}\phi(x,y) = \hat{x}\hat{y}\psi(x,y) . \equiv : \phi(x,y) . \equiv_{x,y} . \psi(x,y),$$

*i.e.* two functions of two variables determine the same relation when, and only when, the two functions are formally equivalent.

We have $\qquad \vdash . z\{\hat{x}\hat{y}\phi(x,y)\}w . \equiv . \phi(z,w),$

*i.e.* "$z$ has to $w$ the relation determined by the function $\phi(x, y)$" is equivalent to $\phi(z, w)$;

$$\vdash :. R = \hat{x}\hat{y}\phi(x,y) . \equiv : xRy . \equiv_{x,y} . \phi(x,y),$$

$$\vdash :. R = S . \equiv : xRy . \equiv_{x,y} . xSy,$$

$$\vdash . \hat{x}\hat{y}(xRy) = R,$$

$$\vdash . \{\hat{x}\hat{y}\phi(x,y)\} \epsilon \mathrm{Rel}.$$

* Such a couple has a *sense*, *i.e.* the couple $(x, y)$ is different from the couple $(y, x)$, unless $x = y$. We shall call it a "couple with sense," to distinguish it from the class consisting of $x$ and $y$. It may also be called an *ordered* couple.

These propositions are analogous to those previously given for classes. It results from them that any function of two variables is formally equivalent to some function of the form $xRy$; hence, in extensional functions of two variables, variation of relations can replace variation of functions of two variables.

Both classes and relations have properties analogous to most of those of propositions that result from negation and the logical sum. The *logical product* of two classes $\alpha$ and $\beta$ is their common part, *i.e.* the class of terms which are members of both. This is represented by $\alpha \cap \beta$. Thus we put

$$\alpha \cap \beta = \hat{x}(x \epsilon \alpha \,.\, x \epsilon \beta) \quad \text{Df.}$$

This gives us $\quad \vdash : x \epsilon \alpha \cap \beta \,.\, \equiv \,.\, x \epsilon \alpha \,.\, x \epsilon \beta,$

*i.e.* "$x$ is a member of the logical product of $\alpha$ and $\beta$" is equivalent to the logical product of "$x$ is a member of $\alpha$" and "$x$ is a member of $\beta$."

Similarly the *logical sum* of two classes $\alpha$ and $\beta$ is the class of terms which are members of either; we denote it by $\alpha \cup \beta$. The definition is

$$\alpha \cup \beta = \hat{x}(x \epsilon \alpha \,.\, \mathsf{v} \,.\, x \epsilon \beta) \quad \text{Df,}$$

and the connection with the logical sum of propositions is given by

$$\vdash :. \, x \epsilon \alpha \cup \beta \,.\, \equiv : x \epsilon \alpha \,.\, \mathsf{v} \,.\, x \epsilon \beta.$$

The *negation* of a class $\alpha$ consists of those terms $x$ for which "$x \epsilon \alpha$" can be *significantly and truly* denied. We shall find that there are terms of other types for which "$x \epsilon \alpha$" is neither true nor false, but nonsense. These terms are not members of the negation of $\alpha$.

Thus the *negation* of a class $\alpha$ is the class of terms of suitable type which are not members of it, *i.e.* the class $\hat{x}(x \sim\epsilon \alpha)$. We call this class "$-\alpha$" (read "not-$\alpha$"); thus the definition is

$$-\alpha = \hat{x}(x \sim\epsilon \alpha) \quad \text{Df,}$$

and the connection with the negation of propositions is given by

$$\vdash : x \epsilon - \alpha \,.\, \equiv \,.\, x \sim\epsilon \alpha.$$

In place of implication we have the relation of *inclusion*. A class $\alpha$ is said to be included or contained in a class $\beta$ if all members of $\alpha$ are members of $\beta$, *i.e.* if $x \epsilon \alpha \,.\, \supset_x .\, x \epsilon \beta$. We write "$\alpha \subset \beta$" for "$\alpha$ is contained in $\beta$." Thus we put

$$\alpha \subset \beta \,.\, = : x \epsilon \alpha \,.\, \supset_x .\, x \epsilon \beta \quad \text{Df.}$$

Most of the formulae concerning $p \,.\, q$, $p \mathsf{v} q$, $\sim p$, $p \supset q$ remain true if we substitute $\alpha \cap \beta$, $\alpha \cup \beta$, $-\alpha$, $\alpha \subset \beta$. In place of equivalence, we substitute identity; for "$p \equiv q$" was defined as "$p \supset q \,.\, q \supset p$," but "$\alpha \subset \beta \,.\, \beta \subset \alpha$" gives "$x \epsilon \alpha \,.\, \equiv_x .\, x \epsilon \beta$," whence $\alpha = \beta$.

The following are some propositions concerning classes which are analogues of propositions previously given concerning propositions:

$$\vdash . \alpha \cap \beta = -(-\alpha \cup -\beta),$$

*i.e.* the common part of $\alpha$ and $\beta$ is the negation of "not-$\alpha$ or not-$\beta$";

$$\vdash . x \,\epsilon\, (\alpha \cup -\alpha),$$

*i.e.* "$x$ is a member of $\alpha$ or not-$\alpha$";

$$\vdash . x \sim\epsilon\, (\alpha \cap -\alpha),$$

*i.e.* "$x$ is not a member of both $\alpha$ and not-$\alpha$";

$$\vdash . \alpha = -(-\alpha),$$

$$\vdash : \alpha \subset \beta . \equiv . -\beta \subset -\alpha,$$

$$\vdash : \alpha = \beta . \equiv . -\alpha = -\beta,$$

$$\vdash : \alpha = \alpha \cap \alpha,$$

$$\vdash : \alpha = \alpha \cup \alpha.$$

The two last are the two forms of the law of tautology.

The law of absorption holds in the form

$$\vdash : \alpha \subset \beta . \equiv . \alpha = \alpha \cap \beta.$$

Thus for example "all Cretans are liars" is equivalent to "Cretans are identical with lying Cretans."

Just as we have $\quad \vdash : p \supset q . q \supset r . \supset . p \supset r,$

so we have $\quad \vdash : \alpha \subset \beta . \beta \subset \gamma . \supset . \alpha \subset \gamma.$

This expresses the ordinary syllogism in Barbara (with the premisses interchanged); for "$\alpha \subset \beta$" means the same as "all $\alpha$'s are $\beta$'s," so that the above proposition states: "If all $\alpha$'s are $\beta$'s, and all $\beta$'s are $\gamma$'s, then all $\alpha$'s are $\gamma$'s." (It should be observed that syllogisms are traditionally expressed with "therefore," as if they asserted both premisses and conclusion. This is, of course, merely a slipshod way of speaking, since what is really asserted is only the connection of premisses with conclusion.)

The syllogism in Barbara when the minor premiss has an individual subject is

$$\vdash : x \,\epsilon\, \beta . \beta \subset \gamma . \supset . x \,\epsilon\, \gamma,$$

*e.g.* "if Socrates is a man, and all men are mortals, then Socrates is a mortal." This, as was pointed out by Peano, is not a particular case of "$\alpha \subset \beta . \beta \subset \gamma . \supset . \alpha \subset \gamma$," since "$x \,\epsilon\, \beta$" is not a particular case of "$\alpha \subset \beta$." This point is important, since traditional logic is here mistaken. The nature and magnitude of its mistake will become clearer at a later stage.

For relations, we have precisely analogous definitions and propositions. We put

$$R \mathbin{\dot\cap} S = \hat{x}\hat{y}\,(xRy . xSy) \quad \text{Df},$$

which leads to $\quad \vdash : x\,(R \mathbin{\dot\cap} S)\,y . \equiv . xRy . xSy.$

Similarly $$R \mathbin{\dot{\cup}} S = \hat{x}\hat{y}\,(xRy \,.\, \lor \,.\, xSy) \quad \text{Df},$$

$$\dot{-}R = \hat{x}\hat{y}\,\{\sim(xRy)\} \quad \text{Df},$$

$$R \mathbin{\dot{\subset}} S \,.\, = : xRy \,.\, \supset_{x,y} \,.\, xSy \quad \text{Df}.$$

Generally, when we require analogous but different symbols for relations and for classes, we shall choose for relations the symbol obtained by adding a dot, in some convenient position, to the corresponding symbol for classes. (The dot must not be put on the line, since that would cause confusion with the use of dots as brackets.) But such symbols require and receive a special definition in each case.

A class is said to *exist* when it has at least one member: "$\alpha$ exists" is denoted by "$\exists ! \alpha$." Thus we put

$$\exists ! \alpha \,.\, = \,.\, (\exists x) \,.\, x \,\epsilon\, \alpha \quad \text{Df}.$$

The class which has no members is called the "null-class," and is denoted by "$\Lambda$." Any propositional function which is always false determines the null-class. One such function is known to us already, namely "$x$ is not identical with $x$," which we denote by "$x \neq x$." Thus we may use this function for defining $\Lambda$, and put

$$\Lambda = \hat{x}\,(x \neq x) \quad \text{Df}.$$

The class determined by a function which is always true is called the *universal class*, and is represented by V; thus

$$\mathrm{V} = \hat{x}\,(x = x) \quad \text{Df}.$$

Thus $\Lambda$ is the negation of V. We have

$$\vdash \,.\, (x) \,.\, x \,\epsilon\, \mathrm{V},$$

*i.e.* "'$x$ is a member of V' is always true"; and

$$\vdash \,.\, (x) \,.\, x \sim\epsilon\, \Lambda,$$

*i.e.* "'$x$ is a member of $\Lambda$' is always false." Also

$$\vdash : \alpha = \Lambda \,.\, \equiv \,.\, \sim\exists ! \alpha,$$

*i.e.* "$\alpha$ is the null-class" is equivalent to "$\alpha$ does not exist."

For relations we use similar notations. We put

$$\dot{\exists} ! R \,.\, = \,.\, (\exists x, y) \,.\, xRy,$$

*i.e.* "$\dot{\exists} ! R$" means that there is at least one couple $x$, $y$ between which the relation $R$ holds. $\dot{\Lambda}$ will be the relation which never holds, and $\dot{\mathrm{V}}$ the relation which always holds. $\dot{\mathrm{V}}$ is practically never required; $\dot{\Lambda}$ will be the relation $\hat{x}\hat{y}\,(x \neq x \,.\, y \neq y)$. We have

$$\vdash \,.\, (x, y) \,.\, \sim(x \,\dot{\Lambda}\, y),$$

and $$\vdash : R = \dot{\Lambda} \,.\, \equiv \,.\, \sim\dot{\exists} ! R.$$

There are no classes which contain objects of more than one type. Accordingly there is a universal class and a null-class proper to each type of object. But these symbols need not be distinguished, since it will be found that there is no possibility of confusion. Similar remarks apply to relations.

*Descriptions.* By a "description" we mean a phrase of the form "*the* so-and-so" or of some equivalent form. For the present, we confine our attention to *the* in the singular. We shall use this word strictly, so as to imply uniqueness; *e.g.* we should not say "$A$ is *the* son of $B$" if $B$ had other sons besides $A$. Thus a description of the form "the so-and-so" will only have an application in the event of there being one so-and-so and no more. Hence a description requires some propositional function $\phi\hat{x}$ which is satisfied by one value of $x$ and by no other values; then "the $x$ which satisfies $\phi\hat{x}$" is a description which definitely describes a certain object, though we may not know what object it describes. For example, if $y$ is a man, "$x$ is the father of $y$" must be true for one, and only one, value of $x$. Hence "the father of $y$" is a description of a certain man, though we may not know *what* man it describes. A phrase containing "the" always presupposes some initial propositional function not containing "the"; thus instead of "$x$ is the father of $y$" we ought to take as our initial function "$x$ begot $y$"; then "the father of $y$" means the one value of $x$ which satisfies this propositional function.

If $\phi\hat{x}$ is a propositional function, the symbol "$(\imath x)(\phi x)$" is used in our symbolism in such a way that it can always be read as "the $x$ which satisfies $\phi\hat{x}$." But we do not define "$(\imath x)(\phi x)$" as standing for "the $x$ which satisfies $\phi\hat{x}$," thus treating this last phrase as embodying a primitive idea. Every use of "$(\imath x)(\phi x)$," where it apparently occurs as a constituent of a proposition in the place of an object, is defined in terms of the primitive ideas already on hand. An example of this definition in use is given by the proposition "$E!\,(\imath x)(\phi x)$" which is considered immediately. The whole subject is treated more fully in Chapter III.

The symbol should be compared and contrasted with "$\hat{x}(\phi x)$" which in use can always be read as "the $x$'s which satisfy $\phi\hat{x}$." Both symbols are incomplete symbols defined only in use, and as such are discussed in Chapter III. The symbol "$\hat{x}(\phi x)$" always has an application, namely to the class determined by $\phi x$; but "$(\imath x)(\phi x)$" only has an application when $\phi\hat{x}$ is only satisfied by one value of $x$, neither more nor less. It should also be observed that the meaning given to the symbol by the definition, given immediately below, of $E!\,(\imath x)(\phi x)$ does not presuppose that we know the meaning of "one." This is also characteristic of the definition of any other use of $(\imath x)(\phi x)$.

We now proceed to define "$E!(\imath x)(\phi x)$" so that it can be read "the $x$ satisfying $\phi x$ exists." (It will be observed that this is a different meaning of existence from that which we express by "$\exists$.") Its definition is

$$E!(\imath x)(\phi x) . = : (\exists c) : \phi x . \equiv_x . x = c \quad \text{Df},$$

*i.e.* "the $x$ satisfying $\phi\hat{x}$ exists" is to mean "there is an object $c$ such that $\phi x$ is true when $x$ is $c$ but not otherwise."

The following are equivalent forms:

$$\vdash :. E!(\imath x)(\phi x) . \equiv : (\exists c) : \phi c : \phi x . \supset_x . x = c,$$

$$\vdash :. E!(\imath x)(\phi x) . \equiv : (\exists c) . \phi c : \phi x . \phi y . \supset_{x,y} . x = y,$$

$$\vdash :. E!(\imath x)(\phi x) . \equiv : (\exists c) : \phi c : x \neq c . \supset_x . \sim\phi x.$$

The last of these states that "the $x$ satisfying $\phi\hat{x}$ exists" is equivalent to "there is an object $c$ satisfying $\phi\hat{x}$, and every object other than $c$ does not satisfy $\phi\hat{x}$."

The kind of existence just defined covers a great many cases. Thus for example "the most perfect Being exists" will mean:

$$(\exists c) : x \text{ is most perfect} . \equiv_x . x = c,$$

which, taking the last of the above equivalences, is equivalent to

$$(\exists c) : c \text{ is most perfect} : x \neq c . \supset_x . x \text{ is not most perfect}.$$

A proposition such as "Apollo exists" is really of the same logical form, although it does not explicitly contain the word *the.* For "Apollo" means really "the object having such-and-such properties," say "the object having the properties enumerated in the Classical Dictionary*." If these properties make up the propositional function $\phi x$, then "Apollo" means "$(\imath x)(\phi x)$," and "Apollo exists" means "$E!(\imath x)(\phi x)$." To take another illustration, "the author of Waverley" means "the man who (or rather, the object which) wrote Waverley." Thus "Scott is the author of Waverley" is

$$\text{Scott} = (\imath x)(x \text{ wrote Waverley}).$$

Here (as we observed before) the importance of *identity* in connection with descriptions plainly appears.

The notation "$(\imath x)(\phi x)$," which is long and inconvenient, is seldom used, being chiefly required to lead up to another notation, namely "$R`y$," meaning "the object having the relation $R$ to $y$." That is, we put

$$R`y = (\imath x)(xRy) \quad \text{Df}.$$

The inverted comma may be read "of." Thus "$R`y$" is read "the $R$ of $y$." Thus if $R$ is the relation of father to son, "$R`y$" means "the father of $y$"; if $R$ is the relation of son to father, "$R`y$" means "the son of $y$," which will

* The same principle applies to many uses of the proper names of existent objects, *e.g.* to all uses of proper names for objects known to the speaker only by report, and not by personal acquaintance.

only "exist" if $y$ has one son and no more. $R$'$y$ is a function of $y$, but not a propositional function; we shall call it a *descriptive* function. All the ordinary functions of mathematics are of this kind, as will appear more fully in the sequel. Thus in our notation, "sin $y$" would be written "sin '$y$," and "sin" would stand for the relation which sin '$y$ has to $y$. Instead of a variable descriptive function $fy$, we put $R$'$y$, where the variable relation $R$ takes the place of the variable function $f$. A descriptive function will in general exist while $y$ belongs to a certain domain, but not outside that domain; thus if we are dealing with positive rationals, $\sqrt{y}$ will be significant if $y$ is a perfect square, but not otherwise; if we are dealing with real numbers, and agree that "$\sqrt{y}$" is to mean the *positive* square root (or, is to mean the negative square root), $\sqrt{y}$ will be significant provided $y$ is positive, but not otherwise; and so on. Thus every descriptive function has what we may call a "domain of definition" or a "domain of existence," which may be thus defined: If the function in question is $R$'$y$, its domain of definition or of existence will be the class of those arguments $y$ for which we have E! $R$'$y$, *i.e.* for which E! $(\imath x)(xRy)$, *i.e.* for which there is one $x$, and no more, having the relation $R$ to $y$.

If $R$ is any relation, we will speak of $R$'$y$ as the "associated descriptive function." A great many of the constant relations which we shall have occasion to introduce are only or chiefly important on account of their associated descriptive functions. In such cases, it is easier (though less correct) to begin by assigning the meaning of the descriptive function, and to deduce the meaning of the relation from that of the descriptive function. This will be done in the following explanations of notation.

*Various descriptive functions of relations.* If $R$ is any relation, the *converse* of $R$ is the relation which holds between $y$ and $x$ whenever $R$ holds between $x$ and $y$. Thus *greater* is the converse of *less*, *before* of *after*, *cause* of *effect*, *husband* of *wife*, etc. The converse of $R$ is written* Cnv'$R$ or $\breve{R}$. The definition is

$$\breve{R} = \hat{x}\hat{y}\,(yRx) \quad \text{Df},$$

$$\text{Cnv}\text{'}R = \breve{R} \quad \text{Df}.$$

The second of these is not a formally correct definition, since we ought to define "Cnv" and deduce the meaning of Cnv'$R$. But it is not worth while to adopt this plan in our present introductory account, which aims at simplicity rather than formal correctness.

A relation is called *symmetrical* if $R = \breve{R}$, *i.e.* if it holds between $y$ and $x$ whenever it holds between $x$ and $y$ (and therefore vice versa). Identity,

* The second of these notations is taken from Schröder's *Algebra und Logik der Relative*.

diversity, agreement or disagreement in any respect, are symmetrical relations. A relation is called *asymmetrical* when it is incompatible with its converse, *i.e.* when $R \dot{\cap} \breve{R} = \dot{\Lambda}$, or, what is equivalent,

$$xRy \, . \supset_{x,y} . \sim (yRx).$$

Before and after, greater and less, ancestor and descendant, are asymmetrical, as are all other relations of the sort that lead to *series*. But there are many asymmetrical relations which do not lead to series, for instance, that of wife's brother*. A relation may be neither symmetrical nor asymmetrical; for example, this holds of the relation of inclusion between classes: $\alpha \subset \beta$ and $\beta \subset \alpha$ will both be true if $\alpha = \beta$, but otherwise only one of them, at most, will be true. The relation *brother* is neither symmetrical nor asymmetrical, for if $x$ is the brother of $y$, $y$ may be either the brother or the sister of $x$.

In the propositional function $xRy$, we call $x$ the *referent* and $y$ the *relatum*. The class $\hat{x}(xRy)$, consisting of all the $x$'s which have the relation $R$ to $y$, is called the class of referents of $y$ with respect to $x$; the class R
$\hat{y}(xRy)$, consisting of all the $y$'s to which $x$ has the relation $R$, is called the class of relata of $x$ with respect to $R$. These two classes are denoted respectively by $\overrightarrow{R}\text{‘}y$ and $\overleftarrow{R}\text{‘}x$. Thus

$$\overrightarrow{R}\text{‘}y = \hat{x}(xRy) \quad \text{Df},$$

$$\overleftarrow{R}\text{‘}x = \hat{y}(yRx) \quad \text{Df}.$$

The arrow runs towards $y$ in the first case, to show that we are concerned with things having the relation $R$ *to* $y$; it runs away from $x$ in the second case to show that the relation $R$ goes *from* $x$ to the members of $\overleftarrow{R}\text{‘}x$. It runs in fact *from* a referent and *towards* a relatum.

The notations $\overrightarrow{R}\text{‘}y$, $\overleftarrow{R}\text{‘}x$ are very important, and are used constantly. If $R$ is the relation of parent to child, $\overrightarrow{R}\text{‘}y =$ the parents of $y$, $\overleftarrow{R}\text{‘}x =$ the children of $x$. We have

$$\vdash : x \in \overrightarrow{R}\text{‘}y \, . \equiv . \, xRy$$

and

$$\vdash : y \in \overleftarrow{R}\text{‘}x \, . \equiv . \, xRy.$$

These equivalences are often embodied in common language. For example, we say indiscriminately "$x$ is an inhabitant of London" or "$x$ inhabits London." If we put "$R$" for "inhabits," "$x$ inhabits London" is "$xR$ London," while "$x$ is an inhabitant of London" is "$x \in \overrightarrow{R}\text{‘}$ London."

* This relation is not strictly asymmetrical, but is so except when the wife's brother is also the sister's husband. In the Greek Church the relation is strictly asymmetrical.

Instead of $\overrightarrow{R}$ and $\overleftarrow{R}$ we sometimes use sg'$R$, gs'$R$, where "sg" stands for "sagitta," and "gs" is "sg" backwards. Thus we put

$$\text{sg}\text{'}R = \overrightarrow{R} \quad \text{Df},$$
$$\text{gs}\text{'}R = \overleftarrow{R} \quad \text{Df}.$$

These notations are sometimes more convenient than an arrow when the relation concerned is represented by a combination of letters, instead of a single letter such as $R$. Thus *e.g.* we should write sg'$(R \dot{\frown} S)$, rather than put an arrow over the whole length of $(R \dot{\frown} S)$.

The class of all terms that have the relation $R$ to something or other is called the *domain* of $R$. Thus if $R$ is the relation of parent and child, the domain of $R$ will be the class of parents. We represent the domain of $R$ by "D'$R$." Thus we put

$$\text{D}\text{'}R = \hat{x} \{(\exists y) . xRy\} \quad \text{Df}.$$

Similarly the class of all terms to which something or other has the relation $R$ is called the *converse domain* of $R$; it is the same as the domain of the converse of $R$. The converse domain of $R$ is represented by "ᗡ'$R$"; thus

$$\text{ᗡ}\text{'}R = \hat{y} \{(\exists x) . xRy\} \quad \text{Df}.$$

The sum of the domain and the converse domain is called the *field*, and is represented by $C$'$R$: thus

$$C\text{'}R = \text{D}\text{'}R \cup \text{ᗡ}\text{'}R \quad \text{Df}.$$

The *field* is chiefly important in connection with series. If $R$ is the ordering relation of a series, $C$'$R$ will be the class of terms of the series, D'$R$ will be all the terms except the last (if any), and ᗡ'$R$ will be all the terms except the first (if any). The first term, if it exists, is the only member of D'$R \cap -$ ᗡ'$R$, since it is the only term which is a predecessor but not a follower. Similarly the last term (if any) is the only member of ᗡ'$R \cap -$D'$R$. The condition that a series should have no end is ᗡ'$R \subset$ D'$R$, *i.e.* "every follower is a predecessor"; the condition for no beginning is D'$R \subset$ ᗡ'$R$. These conditions are equivalent respectively to D'$R = C$'$R$ and ᗡ'$R = C$'$R$.

The *relative product* of two relations $R$ and $S$ is the relation which holds between $x$ and $z$ when there is an intermediate term $y$ such that $x$ has the relation $R$ to $y$ and $y$ has the relation $S$ to $z$. The relative product of $R$ and $S$ is represented by $R \mid S$; thus we put

$$R \mid S = \hat{x}\hat{z} \{(\exists y) . xRy . ySz\} \quad \text{Df},$$

whence $\vdash : x(R \mid S) z . \equiv . (\exists y) . xRy . ySz.$

Thus "paternal aunt" is the relative product of *sister* and *father*; "paternal grandmother" is the relative product of *mother* and *father*; "maternal grandfather" is the relative product of *father* and *mother*. The relative product is not commutative, but it obeys the associative law, *i.e.*

$$\vdash . (P \mid Q) \mid R = P \mid (Q \mid R).$$

It also obeys the distributive law with regard to the logical addition of relations, *i.e.* we have

$$\vdash . P \mid (Q \mathbin{\dot\cup} R) = (P \mid Q) \mathbin{\dot\cup} (P \mid R),$$
$$\vdash . (Q \mathbin{\dot\cup} R) \mid P = (Q \mid P) \mathbin{\dot\cup} (Q \mid R).$$

But with regard to the logical *product*, we have only

$$\vdash . P \mid (Q \mathbin{\dot\cap} R) \subset (P \mid Q) \mathbin{\dot\cap} (P \mid R),$$
$$\vdash . (Q \mathbin{\dot\cap} R) \mid P \subset (Q \mid P) \mathbin{\dot\cap} (Q \mid R).$$

The relative product does not obey the law of tautology, *i.e.* we do not have in general $R \mid R = R$. We put

$$R^2 = R \mid R \quad \text{Df.}$$

Thus paternal grandfather = (father)$^2$,

maternal grandmother = (mother)$^2$.

A relation is called *transitive* when $R^2 \subset R$, *i.e.* when, if $xRy$ and $yRz$, we always have $xRz$, *i.e.* when

$$xRy . yRz . \supset_{x,y,z} . xRz.$$

Relations which generate series are always transitive; thus *e.g.*

$$x > y . y > z . \supset_{x,y,z} . x > z.$$

If $P$ is a relation which generates a series, $P$ may conveniently be read "precedes"; thus "$xPy . yPz . \supset_{x,y,z} . xPz$" becomes "if $x$ precedes $y$ and $y$ precedes $z$, then $x$ always precedes $z$." The class of relations which generate series are partially characterized by the fact that they are transitive and asymmetrical, and never relate a term to itself.

If $P$ is a relation which generates a series, and if we have not merely $P^2 \subset P$, but $P^2 = P$, then $P$ generates a series which is *compact* (*überall dicht*), *i.e.* such that there are terms between any two. For in this case we have

$$xPz . \supset . (\exists y) . xPy . yPz,$$

*i.e.* if $x$ precedes $z$, there is a term $y$ such that $x$ precedes $y$ and $y$ precedes $z$, *i.e.* there is a term between $x$ and $z$. Thus among relations which generate series, those which generate compact series are those for which $P^2 = P$.

Many relations which do not generate series are transitive, for example, identity, or the relation of inclusion between classes. Such cases arise when the relations are not asymmetrical. Relations which are transitive and symmetrical are an important class: they may be regarded as consisting in the possession of some common property.

*Plural descriptive functions.* The class of terms $x$ which have the relation $R$ to some member of a class $\alpha$ is denoted by $R``\alpha$ or $R_\epsilon`\alpha$. The definition is

$$R``\alpha = \hat{x}\{(\exists y) . y \epsilon \alpha . xRy\} \quad \text{Df.}$$

Thus for example let $R$ be the relation of *inhabiting*, and $\alpha$ the class of towns; then $R``\alpha$ = inhabitants of towns. Let $R$ be the relation "less than" among rationals, and $\alpha$ the class of those rationals which are of the form $1 - 2^{-n}$, for integral values of $n$; then $R``\alpha$ will be all rationals less than some member of $\alpha$, *i.e.* all rationals less than 1. If $P$ is the generating relation of a series, and $\alpha$ is any class of members of $P$, $P``\alpha$ will be predecessors of $\alpha$'s, *i.e.* the segment defined by $\alpha$. If $P$ is a relation such that $P`y$ always exists when $y \epsilon \alpha$, $P``\alpha$ will be the class of all terms of the form $P`y$ for values of $y$ which are members of $\alpha$; *i.e.*

$$P``\alpha = \hat{x}\{(\exists y) . y \epsilon \alpha . x = P`y\}.$$

Thus a member of the class "fathers of great men" will be the father of $y$, where $y$ is some great man. In other cases, this will not hold; for instance, let $P$ be the relation of a number to any number of which it is a factor; then $P``$ (even numbers) = factors of even numbers, but this class is not composed of terms of the form "*the* factor of $x$," where $x$ is an even number, because numbers do not have only one factor apiece.

*Unit classes.* The class whose only member is $x$ might be thought to be identical with $x$, but Peano and Frege have shown that this is not the case. (The reasons why this is not the case will be explained in a preliminary way in Chapter II of the Introduction.) We denote by "$\iota`x$" the class whose only member is $x$: thus

$$\iota`x = \hat{y}(y = x) \quad \text{Df,}$$

*i.e.* "$\iota`x$" means "the class of objects which are identical with $x$."

The class consisting of $x$ and $y$ will be $\iota`x \cup \iota`y$; the class got by adding $x$ to a class $\alpha$ will be $\alpha \cup \iota`x$; the class got by taking away $x$ from a class $\alpha$ will be $\alpha - \iota`x$. (We write $\alpha - \beta$ as an abbreviation for $\alpha \cap -\beta$.)

It will be observed that unit classes have been defined without reference to the number 1; in fact, we use unit classes to define the number 1. This number is defined as the class of unit classes, *i.e.*

$$1 = \hat{\alpha}\{(\exists x) . \alpha = \iota`x\} \quad \text{Df.}$$

This leads to

$$\vdash :. \alpha \epsilon 1 . \equiv : (\exists x) : y \epsilon \alpha . \equiv_y . y = x.$$

From this it appears further that

$$\vdash : \alpha \epsilon 1 . \equiv . E ! (\iota x)(x \epsilon \alpha),$$

whence $$\vdash : \hat{z}(\phi z) \epsilon 1 . \equiv . E ! (\iota x)(\phi x),$$

*i.e.* "$\hat{z}(\phi z)$ is a unit class" is equivalent to "the $x$ satisfying $\phi\hat{x}$ exists."

If $\alpha \epsilon 1$, $\breve{\iota}\text{‘}\alpha$ is the only member of $\alpha$, for the only member of $\alpha$ is the only term to which $\alpha$ has the relation $\iota$. Thus "$\breve{\iota}\text{‘}\alpha$" takes the place of "$(\imath x)(\phi x)$," if $\alpha$ stands for $\hat{z}(\phi z)$. In practice, "$\breve{\iota}\text{‘}\alpha$" is a more convenient notation than "$(\imath x)(\phi x)$," and is generally used instead of "$(\imath x)(\phi x)$."

The above account has explained most of the logical notation employed in the present work. In the applications to various parts of mathematics, other definitions are introduced; but the objects defined by these later definitions belong, for the most part, rather to mathematics than to logic. The reader who has mastered the symbols explained above will find that any later formulae can be deciphered by the help of comparatively few additional definitions.