questions of logical analysis, our chief debt is to Frege. Where we differ from him, it is largely because the contradictions showed that he, in common with all other logicians ancient and modern, had allowed some error to creep into his premisses; but apart from the contradictions, it would have been almost impossible to detect this error. In Arithmetic and the theory of series, our whole work is based on that of Georg Cantor. In Geometry we have had continually before us the writings of v. Staudt, Pasch, Peano, Pieri, and Veblen.

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The second volume is already in the press, and both it and the third will appear as soon as the printing can be completed.

A. N. W.

B. R.

CAMBRIDGE,

November, 1910.

INTRODUCTION.

THE mathematical logic which occupies Part I of the present work has been constructed under the guidance of three different purposes. In the first place, it aims at effecting the greatest possible analysis of the ideas with which it deals and of the processes by which it conducts demonstrations, and at diminishing to the utmost the number of the undefined ideas and undemonstrated propositions (called respectively *primitive* ideas and *primitive* propositions) from which it starts. In the second place, it is framed with a view to the perfectly precise expression, in its symbols, of mathematical propositions: to secure such expression, and to secure it in the simplest and most convenient notation possible, is the chief motive in the choice of topics. In the third place, the system is specially framed to solve the paradoxes which, in recent years, have troubled students of symbolic logic and the theory of aggregates; it is believed that the theory of types, as set forth in what follows, leads both to the avoidance of contradictions, and to the detection of the precise fallacy which has given rise to them.

Of the above three purposes, the first and third often compel us to adopt methods, definitions, and notations which are more complicated or more difficult than they would be if we had the second object alone in view. This applies especially to the theory of descriptive expressions (∗14 and ∗30) and to the theory of classes and relations (∗20 and ∗21). On these two points, and to a lesser degree on others, it has been found necessary to make some sacrifice of lucidity to correctness. The sacrifice is, however, in the main only temporary: in each case, the notation ultimately adopted, though its real meaning is very complicated, has an apparently simple meaning which, except at certain crucial points, can without danger be substituted in thought for the real meaning. It is therefore convenient, in a preliminary explanation of the notation, to treat these apparently simple meanings as primitive ideas, *i.e.* as ideas introduced without definition. When the notation has grown more or less familiar, it is easier to follow the more complicated explanations which we believe to be more correct. In the body of the work, where it is necessary to adhere rigidly to the strict logical order the easier order of development could not be adopted; it is therefore given in the Introduction. The explanations given in Chapter I of the Introduction are such as place lucidity before correctness; the full explanations are partly supplied in succeeding Chapters of the Introduction, partly given in the body of the work.

The use of a symbolism, other than that of words, in all parts of the book which aim at embodying strictly accurate demonstrative reasoning, has been forced on us by the consistent pursuit of the above three purposes. The reasons for this extension of symbolism beyond the familiar regions of number and allied ideas are many:

(1) The ideas here employed are more abstract than those familiarly considered in language. Accordingly there are no words which are used mainly in the exact consistent senses which are required here. Any use of words would require unnatural limitations to their ordinary meanings, which would be in fact more difficult to remember consistently than are the definitions of entirely new symbols.

(2) The grammatical structure of language is adapted to a wide variety of usages. Thus it possesses no unique simplicity in representing the few simple, though highly abstract, processes and ideas arising in the deductive trains of reasoning employed here. In fact the very abstract simplicity of the ideas of this work defeats language. Language can represent complex ideas more easily. The proposition "a whale is big" represents language at its best, giving terse expression to a complicated fact; while the true analysis of "one is a number" leads, in language, to an intolerable prolixity. Accordingly terseness is gained by using a symbolism especially designed to represent the ideas and processes of deduction which occur in this work.

(3) The adaptation of the rules of the symbolism to the processes of deduction aids the intuition in regions too abstract for the imagination readily to present to the mind the true relation between the ideas employed. For various collocations of symbols become familiar as representing important collocations of ideas; and in turn the possible relations—according to the rules of the symbolism—between these collocations of symbols become familiar, and these further collocations represent still more complicated relations between the abstract ideas. And thus the mind is finally led to construct trains of reasoning in regions of thought in which the imagination would be entirely unable to sustain itself without symbolic help. Ordinary language yields no such help. Its grammatical structure does not represent uniquely the relations between the ideas involved. Thus, "a whale is big" and "one is a number" both look alike, so that the eye gives no help to the imagination.

(4) The terseness of the symbolism enables a whole proposition to be represented to the eyesight as one whole, or at most in two or three parts divided where the natural breaks, represented in the symbolism, occur. This is a humble property, but is in fact very important in connection with the advantages enumerated under the heading (3).

(5) The attainment of the first-mentioned object of this work, namely the complete enumeration of all the ideas and steps in reasoning employed in mathematics, necessitates both terseness and the presentation of each proposition with the maximum of formality in a form as characteristic of itself as possible.

Further light on the methods and symbolism of this book is thrown by a slight consideration of the limits to their useful employment:

(*a*) Most mathematical investigation is concerned not with the analysis of the complete process of reasoning, but with the presentation of such an abstract of the proof as is sufficient to convince a properly instructed mind. For such investigations the detailed presentation of the steps in reasoning is of course unnecessary, provided that the detail is carried far enough to guard against error. In this connection it may be remembered that the investigations of Weierstrass and others of the same school have shown that, even in the common topics of mathematical thought, much more detail is necessary than previous generations of mathematicians had anticipated.

(β) In proportion as the imagination works easily in any region of thought, symbolism (except for the express purpose of analysis) becomes only necessary as a convenient shorthand writing to register results obtained without its help. It is a subsidiary object of this work to show that, with the aid of symbolism, deductive reasoning can be extended to regions of thought not usually supposed amenable to mathematical treatment. And until the ideas of such branches of knowledge have become more familiar, the detailed type of reasoning, which is also required for the analysis of the steps, is appropriate to the investigation of the general truths concerning these subjects.

CHAPTER I.

PRELIMINARY EXPLANATIONS OF IDEAS AND NOTATIONS.

THE notation adopted in the present work is based upon that of Peano, and the following explanations are to some extent modelled on those which he prefixes to his *Formulario Mathematico.* His use of dots as brackets is adopted, and so are many of his symbols.

Variables. The idea of a variable, as it occurs in the present work, is more general than that which is explicitly used in ordinary mathematics. In ordinary mathematics, a variable generally stands for an undetermined number or quantity. In mathematical logic, any symbol whose meaning is not determinate is called a *variable,* and the various determinations of which its meaning is susceptible are called the *values* of the variable. The values may be any set of entities, propositions, functions, classes or relations, according to circumstances. If a statement is made about "Mr A and Mr B," "Mr A" and "Mr B" are variables whose values are confined to men. A variable may either have a conventionally-assigned range of values, or may (in the absence of any indication of the range of values) have as the range of its values all determinations which render the statement in which it occurs significant. Thus when a text-book of logic asserts that "A is A," without any indication as to what A may be, what is meant is that *any* statement of the form "A is A" is true. We may call a variable *restricted* when its values are confined to some only of those of which it is capable; otherwise, we shall call it *unrestricted.* Thus when an unrestricted variable occurs, it represents any object such that the statement concerned can be made significantly (*i.e.* either truly or falsely) concerning that object. For the purposes of logic, the unrestricted variable is more convenient than the restricted variable, and we shall always employ it. We shall find that the unrestricted variable is still subject to limitations imposed by the manner of its occurrence, *i.e.* things which can be said significantly concerning a proposition cannot be said significantly concerning a class or a relation, and so on. But the limitations to which the unrestricted variable is subject do not need to be explicitly indicated, since they are the limits of significance of the statement in which the variable occurs, and are therefore intrinsically determined by this statement. This will be more fully explained later*.

* Cf. Chapter II of the Introduction.

To sum up, the three salient facts connected with the use of the variable are: (1) that a variable is ambiguous in its denotation and accordingly undefined: (2) that a variable preserves a recognizable identity in various occurrences throughout the same context, so that many variables can occur together in the same context each with its separate identity: and (3) that either the range of possible determinations of two variables may be the same, so that a possible determination of one variable is also a possible determination of the other, or the ranges of two variables may be different, so that, if a possible determination of one variable is given to the other, the resulting complete phrase is meaningless instead of becoming a complete unambiguous proposition (true or false) as would be the case if all variables in it had been given any *suitable* determinations.

The uses of various letters. Variables will be denoted by single letters, and so will certain constants; but a letter which has once been assigned to a constant by a definition must not afterwards be used to denote a variable. The small letters of the ordinary alphabet will all be used for variables, except p and s after $*40$, in which constant meanings are assigned to these two letters. The following capital letters will receive constant meanings: B, C, D, E, F, I and J. Among small Greek letters, we shall give constant meanings to ϵ, ι, ~~π~~ and (at a later stage) to η, θ and ω. Certain Greek capitals will from time to time be introduced for constants, but Greek capitals will not be used for variables. Of the remaining letters, p, q, r will be called *propositional letters*, and will stand for variable propositions (except that, from $*40$ onwards, p must not be used for a variable); f, g, ϕ, ψ, χ, θ and (until $*33$) F will be called *functional letters*, and will be used for variable functions.

The small Greek letters not already mentioned will be used for variables whose values are classes, and will be referred to simply as *Greek letters*. Ordinary capital letters not already mentioned will be used for variables whose values are relations, and will be referred to simply as *capital letters*. Ordinary small letters other than p, q, r, s, f, g will be used for variables whose values are not known to be functions, classes, or relations; these letters will be referred to simply as *small Latin letters*.

After the early part of the work, variable propositions and variable functions will hardly ever occur. We shall then have three main kinds of variables: variable classes, denoted by small Greek letters; variable relations, denoted by capitals; and variables not given as necessarily classes or relations, which will be denoted by small Latin letters.

In addition to this usage of small Greek letters for variable classes, capital letters for variable relations, small Latin letters for variables of type wholly undetermined by the context (these arise from the possibility of "systematic ambiguity," explained later in the explanations of the theory of types), the reader need only remember that all letters represent variables, unless they have been defined as constants in some previous place in the book. In general the structure of the context determines the scope of the variables contained in it; but the special indication of the nature of the variables employed, as here proposed, saves considerable labour of thought.

The fundamental functions of propositions. An aggregation of propositions, considered as wholes not necessarily unambiguously determined, into a single proposition more complex than its constituents, is a function *with propositions as arguments.* The general idea of such an aggregation of propositions, or of variables representing propositions, will not be employed in this work. But there are four special cases which are of fundamental importance, since all the aggregations of subordinate propositions into one complex proposition which occur in the sequel are formed out of them step by step.

They are (1) the Contradictory Function, (2) the Logical Sum, or Disjunctive Function, (3) the Logical Product, or Conjunctive Function, (4) the Implicative Function. These functions in the sense in which they are required in this work are not all independent; and if two of them are taken as primitive undefined ideas, the other two can be defined in terms of them. It is to some extent—though not entirely—arbitrary as to which functions are taken as primitive. Simplicity of primitive ideas and symmetry of treatment seem to be gained by taking the first two functions as primitive ideas.

The Contradictory Function with argument p, where p is any proposition, is the proposition which is the contradictory of p, that is, the proposition asserting that p is not true. This is denoted by $\sim p$. Thus $\sim p$ is the contradictory function with p as argument and means the negation of the proposition p. It will also be referred to as the proposition not-p. Thus $\sim p$ means not-p, which means the negation of p.

The Logical Sum is a propositional function with two arguments p and q, and is the proposition asserting p or q disjunctively, that is, asserting that at least one of the two p and q is true. This is denoted by $p \vee q$. Thus $p \vee q$ is the logical sum with p and q as arguments. It is also called the logical sum of p and q. Accordingly $p \vee q$ means that at least p or q is true, not excluding the case in which both are true.

The Logical Product is a propositional function with two arguments p and q, and is the proposition asserting p and q conjunctively, that is, asserting that both p and q are true. This is denoted by $p \,.\, q$, or—in order to make the dots act as brackets in a way to be explained immediately—by $p : q$, or by $p :. q$, or by $p :: q$. Thus $p \,.\, q$ is the logical product with

p and q as arguments. It is also called the logical product of p and q. Accordingly $p . q$ means that both p and q are true. It is easily seen that this function can be defined in terms of the two preceding functions. For when p and q are both true it must be false that either $\sim p$ or $\sim q$ is true. Hence in this book $p . q$ is merely a shortened form of symbolism for

$$\sim(\sim p \vee \sim q).$$

If any further idea attaches to the proposition "both p and q are true," it is not required here.

The Implicative Function is a propositional function with two arguments p and q, and is the proposition that either not-p or q is true, that is, it is the proposition $\sim p \vee q$. Thus if p is true, $\sim p$ is false, and accordingly the only alternative left by the proposition $\sim p \vee q$ is that q is true. In other words if p and $\sim p \vee q$ are both true, then q is true. In this sense the proposition $\sim p \vee q$ will be quoted as stating that p implies q. The idea contained in this propositional function is so important that it requires a symbolism which with direct simplicity represents the proposition as connecting p and q without the intervention of $\sim p$. But "implies" as used here expresses nothing else than the connection between p and q also expressed by the disjunction "not-p or q." The symbol employed for "p implies q," *i.e.* for "$\sim p \vee q$," is "$p \supset q$." This symbol may also be read "if p, then q." The association of implication with the use of an apparent variable produces an extension called "formal implication." This is explained later: it is an idea derivative from "implication" as here defined. When it is necessary explicitly to discriminate "implication" from "formal implication," it is called "material implication." Thus "material implication" is simply "implication" as here defined. The process of inference, which in common usage is often confused with implication, is explained immediately.

These four functions of propositions are the fundamental constant (*i.e.* definite) propositional functions with *propositions as arguments*, and all other constant propositional functions with propositions as arguments, so far as they are required in the present work, are formed out of them by successive steps. No *variable* propositional functions of this kind occur in this work.

Equivalence. The simplest example of the formation of a more complex function of propositions by the use of these four fundamental forms is furnished by "equivalence." Two propositions p and q are said to be "equivalent" when p implies q and q implies p. This relation between p and q is denoted by "$p \equiv q$." Thus "$p \equiv q$" stands for "$(p \supset q) . (q \supset p)$." It is easily seen that two propositions are equivalent when, and only when, they are both true or are both false. Equivalence rises in the scale of importance when we come to "formal implication" and thus to "formal equivalence." It must not be supposed that two propositions which are equivalent are in any sense identical or even remotely concerned with the same topic. Thus "Newton was a man" and "the sun is hot" are equivalent as being both true, and "Newton was not a man" and "the sun is cold" are equivalent as being both false. But here we have anticipated deductions which follow later from our formal reasoning. Equivalence in its origin is merely mutual implication as stated above.

Truth-values. The "truth-value" of a proposition is *truth* if it is true, and *falsehood* if it is false*. It will be observed that the truth-values of $p \vee q$, $p \,.\, q$, $p \supset q$, $\sim p$, $p \equiv q$ depend only upon those of p and q, namely the truth-value of "$p \vee q$" is truth if the truth-value of either p or q is truth, and is falsehood otherwise; that of "$p \,.\, q$" is truth if that of both p and q is truth, and is falsehood otherwise; that of "$p \supset q$" is truth if either that of p is falsehood or that of q is truth; that of "$\sim p$" is the opposite of that of p; and that of "$p \equiv q$" is truth if p and q have the same truth-value, and is falsehood otherwise. Now the only ways in which propositions will occur in the present work are ways derived from the above by combinations and repetitions. Hence it is easy to see (though it cannot be formally proved except in each particular case) that if a proposition p occurs in any proposition $f(p)$ which we shall ever have occasion to deal with, the truth-value of $f(p)$ will depend, not upon the particular proposition p, but only upon its truth-value; *i.e.* if $p \equiv q$, we shall have $f(p) \equiv f(q)$. Thus whenever two propositions are known to be equivalent, either may be substituted for the other in any formula with which we shall have occasion to deal.

We may call a function $f(p)$ a "truth-function" when its argument p is a proposition, and the truth-value of $f(p)$ depends only upon the truth-value of p. Such functions are by no means the only common functions of propositions. For example, "A believes p" is a function of p which will vary its truth-value for different arguments having the same truth-value: A may believe one true proposition without believing another, and may believe one false proposition without believing another. Such functions are not excluded from our consideration, and are included in the scope of any general propositions we may make about functions; but the particular functions of propositions which we shall have occasion to construct or to consider explicitly are all truth-functions. This fact is closely connected with a characteristic of mathematics, namely, that mathematics is always concerned with extensions rather than intensions. The connection, if not now obvious, will become more so when we have considered the theory of classes and relations.

Assertion-sign. The sign "$\vdash$," called the "assertion-sign," means that what follows is asserted. It is required for distinguishing a complete proposition, which we assert, from any subordinate propositions contained in it but

* This phrase is due to Frege.

not asserted. In ordinary written language a sentence contained between full stops denotes an asserted proposition, and if it is false the book is in error. The sign "$\vdash$" prefixed to a proposition serves this same purpose in our symbolism. For example, if "$\vdash (p \supset p)$" occurs, it is to be taken as a complete assertion convicting the authors of error unless the proposition "$p \supset p$" is true (as it is). Also a proposition stated in symbols without this sign "$\vdash$" prefixed is not asserted, and is merely put forward for consideration, or as a subordinate part of an asserted proposition.

Inference. The process of inference is as follows: a proposition "p" is asserted, and a proposition "p implies q" is asserted, and then as a sequel the proposition "q" is asserted. The trust in inference is the belief that if the two former assertions are not in error, the final assertion is not in error. Accordingly whenever, in symbols, where p and q have of course special determinations,

$$\text{"}\vdash p\text{" and "}\vdash (p \supset q)\text{"}$$

have occurred, then "$\vdash q$" will occur if it is desired to put it on record. The process of the inference cannot be reduced to symbols. Its sole record is the occurrence of "$\vdash q$." It is of course convenient, even at the risk of repetition, to write "$\vdash p$" and "$\vdash (p \supset q)$" in close juxtaposition before proceeding to "$\vdash q$" as the result of an inference. When this is to be done, for the sake of drawing attention to the inference which is being made, we shall write instead

$$\text{"}\vdash p \supset \vdash q,\text{"}$$

which is to be considered as a mere abbreviation of the threefold statement

$$\text{"}\vdash p\text{" and "}\vdash (p \supset q)\text{" and "}\vdash q.\text{"}$$

Thus "$\vdash p \supset \vdash q$" may be read "$p$, therefore q," being in fact the same abbreviation, essentially, as this is; for "p, therefore q" does not explicitly state, what is part of its meaning, that p implies q. An inference is the dropping of a true premiss; it is the dissolution of an implication.

The use of dots. Dots on the line of the symbols have two uses, one to bracket off propositions, the other to indicate the logical product of two propositions. Dots immediately preceded or followed by "$\vee$" or "$\supset$" or "$\equiv$" or "$\vdash$," or by "(x)," "(x, y)," "(x, y, z)"... or "$(\exists x)$," "$(\exists x, y)$," "$(\exists x, y, z)$"... or "$[(\imath x)(\phi x)]$" or "$[R\text{'}y]$" or analogous expressions, serve to bracket off a proposition; dots occurring otherwise serve to mark a logical product. The general principle is that a larger number of dots indicates an outside bracket, a smaller number indicates an inside bracket. The exact rule as to the scope of the bracket indicated by dots is arrived at by dividing the occurrences of dots into three groups which we will name I, II, and III. Group I consists of dots adjoining a sign of implication ($\supset$) or of equivalence ($\equiv$) or of disjunction ($\vee$) or of equality by definition ($=$ Df). Group II consists of dots following brackets indicative of an apparent variable, such as (x) or (x, y) or $(\exists x)$ or

$(\exists x, y)$ or $[(\imath x)(\phi x)]$ or analogous expressions*. Group III consists of dots which stand between propositions in order to indicate a logical product. Group I is of greater force than Group II, and Group II than Group III. The scope of the bracket indicated by any collection of dots extends backwards or forwards beyond any *smaller* number of dots, or any *equal* number from a group of less force, until we reach either the end of the asserted proposition or a *greater* number of dots or an *equal* number belonging to a group of equal or superior force. Dots indicating a logical product have a scope which works both backwards and forwards; other dots only work away from the adjacent sign of disjunction, implication, or equivalence, or forward from the adjacent symbol of one of the other kinds enumerated in Group II.

Some examples will serve to illustrate the use of dots.

"$p \vee q . \supset . q \vee p$" means the proposition "'p or q' implies 'q or p.'" When we *assert* this proposition, instead of merely considering it, we write

$$\text{“} \vdash : p \vee q . \supset . q \vee p, \text{”}$$

where the two dots after the assertion-sign show that what is asserted is the whole of what follows the assertion-sign, since there are not as many as two dots anywhere else. If we had written "$p : \vee : q . \supset . q \vee p$," that would mean the proposition "either p is true, or q implies 'q or p.'" If we wished to assert this, we should have to put three dots after the assertion-sign. If we had written "$p \vee q . \supset . q : \vee : p$," that would mean the proposition "either 'p or q' implies q, or p is true." The forms "$p . \vee . q . \supset . q \vee p$" and "$p \vee q . \supset . q . \vee . p$" have no meaning.

"$p \supset q . \supset : q \supset r . \supset . p \supset r$" will mean "if p implies q, then if q implies r, p implies r." If we wish to assert this (which is true) we write

$$\text{“} \vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r. \text{”}$$

Again "$p \supset q . \supset . q \supset r : \supset . p \supset r$" will mean "if '$p$ implies q' implies 'q implies r,' then p implies r." This is in general untrue. (Observe that "$p \supset q$" is sometimes most conveniently read as "p implies q," and sometimes as "if p, then q.") "$p \supset q . q \supset r . \supset . p \supset r$" will mean "if p implies q, and q implies r, then p implies r." In this formula, the first dot indicates a logical product; hence the scope of the second dot extends backwards to the beginning of the proposition. "$p \supset q : q \supset r . \supset . p \supset r$" will mean "$p$ implies q; and if q implies r, then p implies r." (This is not true in general.) Here the two dots indicate a logical product; since two dots do not occur anywhere else, the scope of these two dots extends backwards to the beginning of the proposition, and forwards to the end.

"$p \vee q . \supset :. p . \vee . q \supset r : \supset . p \vee r$" will mean "if either p or q is true, then if either p or 'q implies r' is true, it follows that either p or r is true."

* The meaning of these expressions will be explained later, and examples of the use of dots in connection with them will be given on pp. 17, 18.

If this is to be asserted, we must put four dots after the assertion-sign, thus:

$$\text{“}\vdash :: p \vee q \,.\supset :.\, p \,.\vee .\, q \supset r : \supset .\, p \vee r.\text{”}$$

(This proposition is proved in the body of the work; it is $*2{\cdot}75$.) If we wish to assert (what is equivalent to the above) the proposition: "if either p or q is true, and either p or 'q implies r' is true, then either p or r is true," we write

$$\text{“}\vdash :.\, p \vee q : p \,.\vee .\, q \supset r : \supset .\, p \vee r.\text{”}$$

Here the first pair of dots indicates a logical product, while the second pair does not. Thus the scope of the second pair of dots passes over the first pair, and back until we reach the three dots after the assertion-sign.

Other uses of dots follow the same principles, and will be explained as they are introduced. In reading a proposition, the dots should be noticed first, as they show its structure. In a proposition containing several signs of implication or equivalence, the one with the greatest number of dots before or after it is the *principal* one: everything that goes before this one is stated by the proposition to imply or be equivalent to everything that comes after it.

Definitions. A definition is a declaration that a certain newly-introduced symbol or combination of symbols is to mean the same as a certain other combination of symbols of which the meaning is already known. Or, if the defining combination of symbols is one which only acquires meaning when combined in a suitable manner with other symbols*, what is meant is that any combination of symbols in which the newly-defined symbol or combination of symbols occurs is to have that meaning (if any) which results from substituting the defining combination of symbols for the newly-defined symbol or combination of symbols wherever the latter occurs. We will give the names of *definiendum* and *definiens* respectively to what is defined and to that which it is defined as meaning. We express a definition by putting the *definiendum* to the left and the *definiens* to the right, with the sign "=" between, and the letters "Df" to the right of the *definiens*. It is to be understood that the sign "=" and the letters "Df" are to be regarded as together forming one symbol. The sign "=" without the letters "Df" will have a different meaning, to be explained shortly.

An example of a definition is

$$p \supset q \,.=.\, \sim p \vee q \quad \text{Df.}$$

It is to be observed that a definition is, strictly speaking, no part of the subject in which it occurs. For a definition is concerned wholly with the symbols, not with what they symbolise. Moreover it is not true or false, being the expression of a volition, not of a proposition. (For this reason,

* This case will be fully considered in Chapter III of the Introduction. It need not further concern us at present.

definitions are not preceded by the assertion-sign.) Theoretically, it is unnecessary ever to give a definition: we might always use the *definiens* instead, and thus wholly dispense with the *definiendum*. Thus although we employ definitions and do not define "definition," yet "definition" does not appear among our primitive ideas, because the definitions are no part of our subject, but are, strictly speaking, mere typographical conveniences. Practically, of course, if we introduced no definitions, our formulae would very soon become so lengthy as to be unmanageable; but theoretically, all definitions are superfluous.

In spite of the fact that definitions are theoretically superfluous, it is nevertheless true that they often convey more important information than is contained in the propositions in which they are used. This arises from two causes. First, a definition usually implies that the *definiens* is worthy of careful consideration. Hence the collection of definitions embodies our choice of subjects and our judgment as to what is most important. Secondly, when what is defined is (as often occurs) something already familiar, such as cardinal or ordinal numbers, the definition contains an analysis of a common idea, and may therefore express a notable advance. Cantor's definition of the continuum illustrates this: his definition amounts to the statement that what he is defining is the object which has the properties commonly associated with the word "continuum," though what precisely constitutes these properties had not before been known. In such cases, a definition is a "making definite": it gives definiteness to an idea which had previously been more or less vague.

For these reasons, it will be found, in what follows, that the definitions are what is most important, and what most deserves the reader's prolonged attention.

Some important remarks must be made respecting the variables occurring in the *definiens* and the *definiendum*. But these will be deferred till the notion of an "apparent variable" has been introduced, when the subject can be considered as a whole.

Summary of preceding statements. There are, in the above, three primitive ideas which are not "defined" but only descriptively explained. Their primitiveness is only relative to our exposition of logical connection and is not absolute; though of course such an exposition gains in importance according to the simplicity of its primitive ideas. These ideas are symbolised by "$\sim p$" and "$p \vee q$," and by "$\vdash$" prefixed to a proposition.

Three definitions have been introduced:

$$p \,.\, q \,.=.\, \sim(\sim p \vee \sim q) \quad \text{Df},$$
$$p \supset q \,.=.\, \sim p \vee q \quad \text{Df},$$
$$p \equiv q \,.=.\, p \supset q \,.\, q \supset p \quad \text{Df}.$$

Primitive propositions. Some propositions must be assumed without proof, since all inference proceeds from propositions previously asserted. These, as far as they concern the functions of propositions mentioned above, will be found stated in ∗1, where the formal and continuous exposition of the subject commences. Such propositions will be called "primitive propositions." These, like the primitive ideas, are to some extent a matter of arbitrary choice; though, as in the previous case, a logical system grows in importance according as the primitive propositions are few and simple. It will be found that owing to the weakness of the imagination in dealing with simple abstract ideas no very great stress can be laid upon their obviousness. They are obvious to the instructed mind, but then so are many propositions which cannot be quite true, as being disproved by their contradictory consequences. The proof of a logical system is its adequacy and its coherence. That is: (1) the system must embrace among its deductions all those propositions which we believe to be true and capable of deduction from logical premisses alone, though possibly they may require some slight limitation in the form of an increased stringency of enunciation; and (2) the system must lead to no contradictions, namely in pursuing our inferences we must never be led to assert both p and not-p, *i.e.* both "$\vdash . p$" and "$\vdash . \sim p$" cannot legitimately appear.

The following are the primitive propositions employed in the calculus of propositions. The letters "Pp" stand for "primitive proposition."

(1) Anything implied by a true premiss is true Pp.

This is the rule which justifies inference.

(2) $\vdash : p \vee p . \supset . p$ Pp,

i.e. if p or p is true, then p is true.

(3) $\vdash : q . \supset . p \vee q$ Pp,

i.e. if q is true, then p or q is true.

(4) $\vdash : p \vee q . \supset . q \vee p$ Pp,

i.e. if p or q is true, then q or p is true.

(5) $\vdash : p \vee (q \vee r) . \supset . q \vee (p \vee r)$ Pp,

i.e. if either p is true or "q or r" is true, then either q is true or "p or r" is true.

(6) $\vdash :. q \supset r . \supset : p \vee q . \supset . p \vee r$ Pp,

i.e. if q implies r, then "p or q" implies "p or r."

(7) Besides the above primitive propositions, we require a primitive proposition called "the axiom of identification of real variables." When we have separately asserted two different functions of x, where x is undetermined, it is often important to know whether we can identify the x in one

assertion with the x in the other. This will be the case—so our axiom states—if both assertions present x as the argument to some one function, that is to say, if ϕx is a constituent in both assertions (whatever propositional function ϕ may be), or, more generally, if $\phi(x, y, z, \ldots)$ is a constituent in one assertion, and $\phi(x, u, v, \ldots)$ is a constituent in the other. This axiom introduces notions which have not yet been explained; for a fuller account, see the remarks accompanying ∗3·03 (which is the statement of this axiom) in the body of the work, as well as the explanation of propositional functions and ambiguous assertion to be given shortly.

Some simple propositions. In addition to the primitive propositions we have already mentioned, the following are among the most important of the elementary properties of propositions appearing among the deductions.

The law of excluded middle:

$$\vdash . p \vee \sim p.$$

This is ∗2·11 below. We shall indicate in brackets the numbers given to the following propositions in the body of the work.

The law of contradiction (∗3·24):

$$\vdash . \sim(p . \sim p).$$

The law of double negation (∗4·13):

$$\vdash . p \equiv \sim(\sim p).$$

The principle of *transposition, i.e.* "if p implies q, then not-q implies not-p," and vice versa: this principle has various forms, namely

(∗4·1) $\vdash : p \supset q . \equiv . \sim q \supset \sim p,$

(∗4·11) $\vdash : p \equiv q . \equiv . \sim p \equiv \sim q,$

(∗4·14) $\vdash :. p . q . \supset . r : \equiv : p . \sim r . \supset . \sim q,$

as well as others which are variants of these.

The law of tautology, in the two forms:

(∗4·24) $\vdash : p . \equiv . p . p,$

(∗4·25) $\vdash : p . \equiv . p \vee p,$

i.e. "p is true" is equivalent to "p is true and p is true," as well as to "p is true or p is true." From a formal point of view, it is through the law of tautology and its consequences that the algebra of logic is chiefly distinguished from ordinary algebra.

The law of absorption:

(∗4·71) $\vdash :. p \supset q . \equiv : p . \equiv . p . q,$

i.e. "p implies q" is equivalent to "p is equivalent to $p . q$." This is called the law of absorption because it shows that the factor q in the product is absorbed by the factor p, if p implies q. This principle enables us to replace an implication ($p \supset q$) by an equivalence ($p . \equiv . p . q$) whenever it is convenient to do so.

An analogous and very important principle is the following:

$$(*4{\cdot}73) \quad \vdash :. q . \supset : p . \equiv . p . q.$$

Logical addition and multiplication of propositions obey the associative and commutative laws, and the distributive law in two forms, namely

$$(*4{\cdot}4) \quad \vdash :. p . q \vee r . \equiv : p . q . \vee . p . r,$$
$$(*4{\cdot}41) \quad \vdash :. p . \vee . q . r : \equiv : p \vee q . p \vee r.$$

The second of these distinguishes the relations of logical addition and multiplication from those of arithmetical addition and multiplication.

Propositional functions. Let ϕx be a statement containing a variable x and such that it becomes a proposition when x is given any fixed determined meaning. Then ϕx is called a "propositional function"; it is not a proposition, since owing to the ambiguity of x it really makes no assertion at all. Thus "x is hurt" really makes no assertion at all, till we have settled who x is. Yet owing to the individuality retained by the ambiguous variable x, it is an ambiguous example from the collection of propositions arrived at by giving all possible determinations to x in "x is hurt" which yield a proposition, true or false. Also if "x is hurt" and "y is hurt" occur *in the same context*, where y is another variable, then according to the determinations given to x and y, they can be settled to be (possibly) the same proposition or (possibly) different propositions. But apart from some determination given to x and y, they retain in that context their ambiguous differentiation. Thus "x is hurt" is an ambiguous "value" of a propositional function. When we wish to speak of the propositional function corresponding to "x is hurt," we shall write "$\hat{x}$ is hurt." Thus "$\hat{x}$ is hurt" is the propositional function and "x is hurt" is an ambiguous value of that function. Accordingly though "x is hurt" and "y is hurt" *occurring in the same context* can be distinguished, "$\hat{x}$ is hurt" and "$\hat{y}$ is hurt" convey no distinction of meaning at all. More generally, ϕx is an ambiguous value of the propositional function $\phi \hat{x}$, and when a definite signification a is substituted for x, ϕa is an unambiguous value of $\phi \hat{x}$.

Propositional functions are the fundamental kind from which the more usual kinds of function, such as "$\sin x$" or "$\log x$" or "the father of x," are derived. These derivative functions are considered later, and are called "descriptive functions." The functions of propositions considered above are a particular case of propositional functions.

The range of values and total variation. Thus corresponding to any propositional function $\phi \hat{x}$, there is a range, or collection, of values, consisting of all the propositions (true or false) which can be obtained by giving every possible determination to x in ϕx. A value of x for which ϕx is true will be said to "satisfy" $\phi \hat{x}$. Now in respect to the truth or falsehood of propositions of this range three important cases must be noted and symbolised. These cases are given by three propositions of which one at least must be true. Either (1) all propositions of the range are true, or (2) some propositions of the range are true, or (3) no proposition of the range is true. The statement (1) is symbolised by "$(x) . \phi x$," and (2) is symbolised by "$(\exists x) . \phi x$." No definition is given of these two symbols, which accordingly embody two new primitive ideas in our system. The symbol "$(x) . \phi x$" may be read "ϕx always," or "ϕx is always true," or "ϕx is true for all possible values of x." The symbol "$(\exists x) . \phi x$" may be read "there exists an x for which ϕx is true," or "there exists an x satisfying $\phi \hat{x}$," and thus conforms to the natural form of the expression of thought.

Proposition (3) can be expressed in terms of the fundamental ideas now on hand. In order to do this, note that "$\sim \phi x$" stands for the contradictory of ϕx. Accordingly $\sim \phi \hat{x}$ is another propositional function such that each value of $\phi \hat{x}$ contradicts a value of $\sim \phi \hat{x}$, and vice versa. Hence "$(x) . \sim \phi x$" symbolises the proposition that every value of $\phi \hat{x}$ is untrue. This is number (3) as stated above.

It is an obvious error, though one easy to commit, to assume that cases (1) and (3) are each other's contradictories. The symbolism exposes this fallacy at once, for (1) is $(x) . \phi x$, and (3) is $(x) . \sim \phi x$, while the contradictory of (1) is $\sim \{(x) . \phi x\}$. For the sake of brevity of symbolism a definition is made, namely

$$\sim (x) . \phi x . = . \sim \{(x) . \phi x\} \quad \text{Df.}$$

Definitions of which the object is to gain some trivial advantage in brevity by a slight adjustment of symbols will be said to be of "merely symbolic import," in contradistinction to those definitions which invite consideration of an important idea.

The proposition $(x) . \phi x$ is called the "total variation" of the function $\phi \hat{x}$.

For reasons which will be explained in Chapter II, we do not take negation as a primitive idea when propositions of the forms $(x) . \phi x$ and $(\exists x) . \phi x$ are concerned, but we *define* the negation of $(x) . \phi x$, *i.e.* of "ϕx is always true," as being "ϕx is sometimes false," *i.e.* "$(\exists x) . \sim \phi x$," and similarly we *define* the negation of $(\exists x) . \phi x$ as being $(x) . \sim \phi x$. Thus we put

$$\sim \{(x) . \phi x\} . = . (\exists x) . \sim \phi x \quad \text{Df,}$$

$$\sim \{(\exists x) . \phi x\} . = . (x) . \sim \phi x \quad \text{Df.}$$

In like manner we define a disjunction in which one of the propositions is of the form "$(x) . \phi x$" or "$(\exists x) . \phi x$" in terms of a disjunction of propositions not of this form, putting

$$(x) . \phi x . \vee . p : = . (x) . \phi x \vee p \quad \text{Df,}$$

i.e. "either ϕx is always true, or p is true" is to mean "'ϕx or p' is always true," with similar definitions in other cases. This subject is resumed in Chapter II, and in $*9$ in the body of the work.

Apparent variables. The symbol "$(x) . \phi x$" denotes one definite proposition, and there is no distinction in meaning between "$(x) . \phi x$" and "$(y) . \phi y$" when they occur in the same context. Thus the "x" in "$(x) . \phi x$" is not an ambiguous constituent of any expression in which "$(x) . \phi x$" occurs; and such an expression does not cease to convey a determinate meaning by reason of the ambiguity of the x in the "ϕx." The symbol "$(x) . \phi x$" has some analogy to the symbol

$$\text{“}\int_a^b \phi(x)\, dx\text{”}$$

for definite integration, since in neither case is the expression a function of x.

The range of x in "$(x) . \phi x$" or "$(\exists x) . \phi x$" extends over the complete field of the values of x for which "ϕx" has meaning, and accordingly the meaning of "$(x) . \phi x$" or "$(\exists x) . \phi x$" involves the supposition that such a field is determinate. The x which occurs in "$(x) . \phi x$" or "$(\exists x) . \phi x$" is called (following Peano) an "apparent variable." It follows from the meaning of "$(\exists x) . \phi x$" that the x in this expression is also an apparent variable. A proposition in which x occurs as an apparent variable is not a function of x. Thus *e.g.* "$(x) . x = x$" will mean "everything is equal to itself." This is an absolute constant, not a function of a variable x. This is why the x is called an *apparent* variable in such cases.

Besides the "*range*" of x in "$(x) . \phi x$" or "$(\exists x) . \phi x$," which is the field of the values that x may have, we shall speak of the "*scope*" of x, meaning the function of which all values or some value are being affirmed. If we are asserting all values (or some value) of "ϕx," "ϕx" is the scope of x; if we are asserting all values (or some value) of "$\phi x \supset p$," "$\phi x \supset p$" is the scope of x; if we are asserting all values (or some value) of "$\phi x \supset \psi x$," "$\phi x \supset \psi x$" will be the scope of x, and so on. The scope of x is indicated by the number of dots after the "(x)" or "$(\exists x)$"; that is to say, the scope extends forwards until we reach an equal number of dots not indicating a logical product, or a greater number indicating a logical product, or the end of the asserted proposition in which the "(x)" or "$(\exists x)$" occurs, whichever of these happens first*. Thus *e.g.*

$$\text{“}(x) : \phi x . \supset . \psi x\text{”}$$

will mean "ϕx always implies ψx," but

$$\text{“}(x) . \phi x . \supset . \psi x\text{”}$$

will mean "if ϕx is always true, then ψx is true for the argument x."

Note that in the proposition

$$(x) . \phi x . \supset . \psi x$$

* This agrees with the rules for the occurrences of dots of the type of Group II as explained above, pp. 9 and 10.

the two x's have no connection with each other. Since only one dot follows the x in brackets, the scope of the first x is limited to the "ϕx" immediately following the x in brackets. It usually conduces to clearness to write

$$(x) . \phi x . \supset . \psi y$$

rather than

$$(x) . \phi x . \supset . \psi x,$$

since the use of different letters emphasises the absence of connection between the two variables; but there is no logical necessity to use different letters, and it is *sometimes* convenient to use the same letter.

Ambiguous assertion and the real variable. Any value "ϕx" of the function $\phi \hat{x}$ can be asserted. Such an assertion of an ambiguous member of the values of $\phi \hat{x}$ is symbolised by

$$\text{"} \vdash . \phi x. \text{"}$$

Ambiguous assertion of this kind is a primitive idea, which cannot be defined in terms of the assertion of propositions. This primitive idea is the one which embodies the use of the variable. Apart from ambiguous assertion, the consideration of "ϕx," which is an ambiguous member of the values of $\phi \hat{x}$, would be of little consequence. When we are considering or asserting "ϕx," the variable x is called a "real variable." Take, for example, the law of excluded middle in the form which it has in traditional formal logic:

"a is either b or not b."

Here a and b are real variables: as they vary, different propositions are expressed, though all of them are true. While a and b are undetermined, as in the above enunciation, no one definite proposition is asserted, but what is asserted is *any* value of the propositional function in question. This can only be legitimately asserted if, whatever value may be chosen, that value is true, *i.e.* if all the values are true. Thus the above form of the law of excluded middle is equivalent to

"$(a, b) . a$ is either b or not b,"

i.e. to "it is always true that a is either b or not b." But these two, though equivalent, are not identical, and we shall find it necessary to keep them distinguished.

When we assert something containing a real variable, as in *e.g.*

$$\text{"} \vdash . x = x, \text{"}$$

we are asserting *any* value of a propositional function. When we assert something containing an apparent variable, as in

$$\text{"} \vdash . (x) . x = x \text{"}$$

or

$$\text{"} \vdash . (\exists x) . x = x, \text{"}$$

we are asserting, in the first case *all* values, in the second case *some* value (undetermined), of the propositional function in question. It is plain that