13.

SUR LES QUANTITÉS FORMANT UN GROUPE DE NONIONS ANALOGUES AUX QUATERNIONS DE HAMILTON.

[Comptes Rendus, xcvii (1883), pp. 1336—1340.]

On sait qu'on peut tout à fait (et très avantageusement) changer la base de la théorie des quaternions en considérant les trois symboles $i$, $j$, $k$ de Hamilton comme des matrices binaires.

Si $h$, $j$ sont des matrices binaires qui satisfont à l'équation $hj = -jh$, on démontre facilement que, en écartant le cas où $hj = jh = 0$, $h^2$ et $k^2$ seront de la forme

\[
\begin{array}{ccc}
  c & 0 & \gamma \\
 0 & c' & 0 \\
\end{array}
\]

$\gamma$ c'est-à-dire $cu$, $\gamma u$, où $u$ est l'unité binaire

\[
\begin{array}{ccc}
  1 & 0 \\
 0 & 1 \\
\end{array}
\]

On peut ajouter, si l'on veut, les deux conditions $c^2 = \bar{I}$, $\gamma^2 = \bar{I}$; alors, en supprimant, pour plus de brièveté, le $u$, qui jouit de propriétés tout à fait analogues à celles de l'unité ordinaire, on obtient facilement les équations connues

\[
h^2 = \bar{I}, \quad j^2 = \bar{I}, \quad k^2 = \bar{I},
\]

\[
hj = -jh = k, \quad jk = -kj = i, \quad ki = -ik = j.
\]

De plus, en supposant que $(i, j)$ soit un système particulier qui satisfaire à l'équation $ij = -ji$, on peut déduire les valeurs universelles de $I$, $J$ qui satisfont à l'équation $IJ = -JI$ en termes de $i$, $j$. En effet, on démontre rigoureusement que, en écartant toujours la solution $mn = nm = 0$, on aura

\[
I = ai + bj + ci, \\
J = ai + bj + ci,
\]

avec la seule condition $a\alpha + b\beta + c\gamma = 0$. De plus, si l'on suppose $\alpha = j^2 = \bar{u}$ et aussi $I = J = \bar{u}$, on aura

\[
a^2 + b^2 + c^2 = 1, \quad a^2 + b^2 + c^2 = 1,
\]

\[
\alpha^2 + \beta^2 + \gamma^2 = 1,
\]

\[
\alpha^2 + \beta^2 + \gamma^2 = 1,
\]
de sorte que, en écrivant \( ij = k, IJ = K \) et \( K = Ai + Bj + Ck \), la matrice
\[
\begin{bmatrix}
a & b & c \\
a & \beta & \gamma \\
A & B & C
\end{bmatrix}
\]
formera une matrice orthogonale. Une solution, parmi les plus simples, des équations \( ij = -ji, i^2 = \bar{u}, j^2 = \bar{u} \), est la suivante :
\[
i = \begin{bmatrix} \theta & 0 \\ 0 & -\theta \end{bmatrix}, \quad j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]
et conséquemment
\[
k = ij = \begin{bmatrix} 0 & -\theta \\ -\theta & 0 \end{bmatrix},
\]
où \( \theta = \sqrt{(-1)} \).

En écrivant une quantité binormale quelconque (c'est-à-dire une matrice binaire) sous la forme
\[
a + b\theta, \quad -c - d\theta,
\]
\[
c - d\theta, \quad a - b\theta,
\]
on voit qu'elle peut être mise sous la forme \( au + bi + cj + dk \), où il est souvent commode de supprimer (c'est-à-dire de sous-entendre) sans écrire l'unité binaire \( u \).

On peut construire d'une manière tout à fait analogue un système de nonions en considérant l'équation \( m = \rho n \), où \( m, n \) sont des matrices ternaires et \( \rho \) une racine cubique primitive de l'unité (voir* la Circular du Johns Hopkins University qui va prochainement paraître), en prenant pour les nonions fondamentaux \( u \) (l'unité ternaire)
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
et les huit matrices \( m, m^2; n, n^2; m^2 n, mn^2; mn, m^2 n^2 \) construites avec les valeurs les plus simples de \( m, n \) qui satisfont aux équations
\[
mn = \rho mn, \quad m^3 = u, \quad n^3 = u.
\]

Les valeurs
\[
m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{bmatrix} \quad \text{et} \quad n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \rho \\ \rho^2 & 0 & 0 \end{bmatrix}
\]
peuvent être prises pour les valeurs basiques du système de nonions.

Une quantité ternaire (c'est-à-dire une matrice) quelconque s'exprime alors sous la forme
\[
a + bm + \beta m^2 + cn + \gamma n^2 + dm^2 n + \delta mn^2 + emn + em^2 n^2;
\]

[* Vol. iii. of this Reprint, p. 647. Also below, p. 122.]
mais, quand cette matrice $M$ est capable de s'associer avec une autre $N$ dans l'équation $NM = ρMN$, alors il devient nécessaire que
\[ a = 0, \quad bβ + cγ + dδ + ee = 0. \]

Je n'entrerai pas ici dans les détails de la méthode d'associer la solution générale de l'équation $NM = ρMN$ avec une solution quelconque particulière de cette équation, mais je me bornerai à expliquer quelles sont les conditions auxquelles les éléments de $M$ et de $N$ doivent satisfaire afin que cette équation ait lieu.

M. Cayley a résolu la question analogue pour les matrices binaires dans le beau Mémoire, qu'il a publié dans les Transactions of the Royal Society de 1858. En supposant que $m$ et $n$ sont les matrices
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
a' & b' \\
c' & d'
\end{pmatrix}
\]
il trouve que, afin que $nm = -mn$, il faut avoir
\[ a + d' = 0, \quad a' + d = 0, \quad aa' + bc' + c'b' + dd' = 0. \]

Au lieu de cette troisième équation (en la combinant avec les deux précédentes), on peut écrire
\[ ad' + a'd - bc' - b'c = 0. \]

Alors ces trois conditions équivalent à dire que le déterminant de la matrice $xu + my + nz$ (u étant l'unité binaire), qui, en général, est de la forme
\[ x^2 + 2Bxy + 2Czx + Dy^2 + 2Eyze + Fz^2, \]
se réduira à la forme
\[ x^2 + Dy^2 + Fz^2, \]

car, dans le déterminant de $xu + my + nz$, c'est-à-dire de
\[
\begin{vmatrix}
x + ay + a'z & by + b'z \\
cy + c'z & x + dy + d'z
\end{vmatrix},
\]
les coefficients de $xy$, $xz$, $yz$ seront évidemment
\[ a + d, \quad a' + d', \quad ad' + a'd - bc' - b'c \]
respectivement.

Passons au cas de $m$ et $n$, matrices ternaires qui satisfont à l'équation
\[ nm = ρmn. \]

Formons le déterminant de $xu + ym + zn$, où $u$ représente l'unité ternaire
\[
\begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{vmatrix}.
\]
Ce déterminant sera de la forme

\[ a^2 + 3Bx^2y + 3Cy^2z + 3Dxy^2 + 6Exyz + 3Fzx^2 + Gy^2 + 3Hy^2z + 3Kyz^2 + Lz^2, \]

et je trouve que, dans le cas supposé, il faut que les sept conditions souscrites soient satisfaites; \( B = 0, \ C = 0, \ D = 0, \ E = 0, \ F = 0, \ H = 0, \ K = 0 \), de sorte que la fonction en \( x, \ y, \ z \) devient une somme de trois cubes, mais ces sept conditions, qu'on pourrait nommer conditions paramétriques, quoique nécessaires, ne sont pas suffisantes; il faut y ajouter une huitième condition que je nommerai \( Q = 0 \).

Pour former \( Q \), voici la manière de procéder:

En supposant que

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & k
\end{vmatrix}
\quad \text{et} \quad
\begin{vmatrix}
  a' & b' & c' \\
  d' & e' & f' \\
  g' & h' & k'
\end{vmatrix},
\]

on écrit, au lieu de \( m \), son transversal

\[
\begin{vmatrix}
  a' & d' & g' \\
  b' & e' & h' \\
  c' & f' & k'
\end{vmatrix},
\]

et l'on forme neuf produits en multipliant chaque déterminant mineur du second ordre contenu dans \( m \) avec le déterminant mineur semblablement posé dans le transversal de \( n \); la somme de ces neuf produits est \( Q \).

Ces huit conditions que je démontre sont suffisantes et nécessaires (en écartant comme auparavant le cas où \( nm = mn = 0 \)) pour que \( nm = pmn \).

On pourrait très bien se demander ce qui arrive dans le cas où les sept conditions paramétriques sont satisfaites, mais non pas la huitième condition supplémentaire.

Dans ce cas, je trouve* que \( mn \) et \( nm \) restent fonctions l'une et l'autre et qu'on aura

\[
\begin{align*}
  nm &= A + B_1mn + C (mn)^2, \\
  mn &= -A + B_2nm + C (nm)^2,
\end{align*}
\]

où \( B_1, \ B_2 \) sont les racines de l'équation algébrique

\[ B^2 + B + 1 = 0, \]

\( A, \ C \) étant deux quantités arbitraires et indépendantes, sauf que l'une d'elles ne peut pas s'évanouir sans l'autre, les deux s'évanouissant ensemble pour le cas (et seulement pour le cas) où \( Q \) (qui fournit la condition supplémentaire) s'évanouit.

[* See footnote [†], p. 154 below.]
ON QUATERNIONS, NONIONS, SEDENIONS, ETC.

[Johns Hopkins University Circulars, III. (1884), pp. 7—9.]

(1) Suppose that $m$ and $n$ are two matrices of the second order.

Then if we call the determinant of the matrix $x + my + nz$, 
\[
x^2 + 2bxy + 2cxy + dy^2 + 2eyz + fz^2,
\]
the necessary and sufficient conditions for the subsistence of the equation $mn = -nm$ is that $b = 0$, $c = 0$, $e = 0$, and if we superadd the equations $m^2 + 1 = 0$, $n^2 + 1 = 0$, then $d = 1$ and $f = 1$, or in other words in order to satisfy the equations $mn = -nm$, $m^2 = -1$, $n^2 = -1$, where it will of course be understood that in these (as in the equations $m^2 + 1 = 0$, $n^2 + 1 = 0$) 1 is the abbreviated form of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\overline{1}$ of the form $\begin{pmatrix} \overline{1} & 0 \\ 0 & \overline{1} \end{pmatrix}$, the necessary and sufficient condition is that the determinant of $x + my + nz$ shall be equal to $x^2 + y^2 + z^2$.

The simplest mode of satisfying this condition is to write $m = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $n = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $i$ meaning $\sqrt{-1}$, which gives $mn = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ and $nm = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

It is easy to express any matrix of the second order as a linear function of $I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} m, n, p$, where $p$ stands for $mn$.

For if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any such matrix it is only necessary to write

\[
a = f + ig, \quad b = -h - ki, \\
d = f - ig, \quad c = -h + ki,
\]

and then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = f + gm + hn + kp$.

The most general solution of the equations $MN = -NM, M^2 = N^2 = -1$, must contain three arbitrary constants, namely, the difference between the number of terms in $m$ and $n$, and the number of conditions $b = 0$, $c = 0$, $e = 0$, $d = 1$, $f = 1$, which are to be satisfied.

[* I denotes $-1$.]
Suppose $M, N$ to be the most general solution fulfilling these conditions; we may write

\[ M = f + gm + hn + kp, \]
\[ N = f' + g'm + h'n + k'p, \]

where $m, n$ is any particular solution and $p = mn$, and we shall have inasmuch as $M^2 = 1$,

\[ f^2 - g^2 - h^2 - k^2 + 2fgm + 2fhn + 2fkp = \text{the matrix } 1, \]

and consequently

\[ g^2 + h^2 + k^2 = 1 + f^2, \]
\[ fg = 0, \quad fh = 0, \quad fk = 0. \]

Hence $f = 0$ and

\[ g^2 + h^2 + k^2 = 1. \]

Similarly $f' = 0$ and

\[ g'^2 + h'^2 + k'^2 = 1, \]

and also inasmuch as $MN = -NM$,

\[ gg' + hh' + kk' = 0, \]

and since the equations $M^2 = 1, N^2 = 1, MN = -NM$ imply if we make $MN = P$ that $P^2 = -1$, and $MP = -PM$, and $NP = -PN$, it follows that $M, N, P$, are connected with $m, n, p, q$, in the same way as the coordinates of a point referred to one set of rectangular coordinates in space are connected with the coordinates of the same point referred to any other set of the same.*

Herein lies the ground of the geometrical interpretation to which quaternions lend themselves and it is hardly necessary to do more than advert to the fact that the theory of Quaternions is one and the same thing as that of Matrices of the second order viewed under a particular aspect †.

(2) Let $m, n$ now denote matrices of the third order.

We might propose to solve the equation $mn = -nm$.

The result of the investigation is that we must have $m^2 = n^2, m^2 = 0, n^2 = 0$, and writing $mn = p, m^2 = n^2 = q$, there results a set of quiniones, $1, m, n, p, q$, for which the multiplication is that marked $(a_s)$ p. 144 of the late Prof. Peirce’s invaluable memoir in Vol. iv. of the American Journal of Mathematics.

* There is another solution possible, obtained by writing

\[ \frac{f}{f'} = \frac{g}{g'} = \frac{h}{h'}, \quad f^2 + g^2 + h^2 + k^2 = 0 \]

but this leads to a linear relation between $m$ and $n$, so that $mn = nm$ and consequently $mn = nm = 0$ which is not the kind of solution proposed in the question.

† See my article in the Lond. and Edin. Phil. Mag. on “Involution and Evolution of Quaternions,” November, 1888. [Above, p. 119.]
But instead of this let us propose the equation $mn = \rho mn$, where $\rho$ is one of the imaginary roots of unity; if now we write the determinant of $x + my + nz$ under the form

$$x^2 + 3bx^2y + 3cx^2z + 3dxz^2 + 6exyz + 3fyz^2 + gy^2 + 3hy^2z + 3kys^2 + lz^2,$$

it may be shown [cf. p. 126, below] that we must have

$$b = 0, \quad c = 0, \quad d = 0, \quad e = 0, \quad f = 0, \quad h = 0, \quad k = 0,$$

and if we superadd the conditions $m^2 = 1$, $n^2 = 1$, we must also have $g = 1$, $l = 1$, or in other words the determinant to $x + my + nz$ must take the form $x^2 + y^2 + z^2$; but this condition (or system of conditions) although necessary is not sufficient (a point which I omitted to notice in my article entitled "A Word on Nonions" inserted* in a previous Circular).

It is obviously necessary that we must have $(mn)^2 = 1$.

Now if the identical equation to $mn$ be written under the form

$$(mn)^2 - 3B(mn)^2 + 3Dmn - E = 0,$$

$B$ may be shown to be a linear homogeneous function of $b, c, \text{ and } e$; also $E = gl = 1$; but $D$ is not a function of $b, c, d, e, f, g, h, k, l$, and will not in general vanish (as it is here required to do) when $b, c, d, e, f, h, k$ vanish. Its value is the sum of the products obtained on multiplying each quadratic minor of $m$ by its altruistic opposite in $n$: (the proper opposite to a minor of $m$ means the minor which is the reflected image of such minor viewed in the Principal Diagonal of $m$ regarded as a mirror; and the altruistic opposite is the minor which occupies in $n$ a position precisely similar to that of the proper opposite in $m$). There are, therefore, 10 equations in all to be satisfied between the coefficients of $m$ and $n$ when $m^2 = n^2 = 1$ and $mn = \rho mn$.

These ten conditions I have demonstrated are sufficient as well as necessary. There remains then $18 - 10 = 8$ arbitrary constants in the general solution. If $m, n$ is a particular solution we may take for $M, N$ (the matrices of the general solution),

$$M = \alpha + \beta m + \gamma m^2 + \alpha' n + \beta' mn + \gamma' m^2 n + \alpha'' n^2 + \beta'' mn^2 + \gamma'' m^2 n^2,$$

$$N = \alpha_i + \beta_i m + \gamma_i m^2 + \alpha_i' n + \beta_i' mn + \gamma_i' m^2 n + \alpha_i'' n^2 + \beta_i'' mn^2 + \gamma_i'' m^2 n^2,$$

and 10 relations between the 18 coefficients must be sufficient to enable to be satisfied the equations $M^2 = N^2 = 1$, $NM = \rho MN$; but what these relations are and how they may most simply be expressed I am not at present in a condition to state†.

*[Vol. iii. of this Reprint, p. 647.]

† The solution of this problem would seem to involve some unknown expansion of the idea of orthogonalism. Unless $MN = NM = 0$, a solution to be neglected, it may be proved that $a=0, a_1=0$.}
I showed in "A Word on Nonions" that the 9 first conditions are satisfied by taking

\[
\begin{align*}
1 & 0 & 0 & 0 & 0 & 1 \\
m & = & 0 & \rho & 0 & n = \rho & 0 & 0 \\
0 & 0 & \rho^2 & 0 & 0 & \rho^2 & 0 & 0.
\end{align*}
\]

The 10th condition is also satisfied; for the only quadratic minors (not having a zero determinant) in \(m\) are \( \begin{bmatrix} 1 & 0 & \rho & 0 & 1 & 0 \\
0 & \rho & 0 & \rho^2 & 0 & \rho^2 \end{bmatrix} \); the altruistic opposites to which in \(n\) are \( \begin{bmatrix} 0 & \rho & 0 & \rho^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \), the determinants to each of which are zeros, and accordingly we find

\[
\begin{align*}
m^2 & = n^2 = 0 & 1 & 0 \\
0 & 0 & \rho^2 & 0 & 0 & 1 \\
mn & = \rho & 0 & 0 & mn & = \rho^2 & 0 & 0 \\
0 & 1 & 0, & 0 & \rho & 0 & 0,
\end{align*}
\]

so that \(mn = nmn\) and \(m^2 = n^2 = 1\) as required.

I subjoin an outline proof of the fundamental portion of the theory of Quaternions and Nonions above stated as it will serve to throw much light upon the nature of the processes employed in that new world of thought to which I gave the name of Universal Algebra or the Algebra of multiple quantity: a fuller explanation will be found in the long memoir which I am preparing on the entire subject for the *American Journal of Mathematics*.

(1) As regards the equation \(nm = -mn\), where \(m, n\) are matrices of the second order.

As before let the determinant of \((x + ym + zn)\) be

\[x^2 + 2bxy + 2cxs + dy^2 + 2eyz + fz^2.\]

I may observe here parenthetically that the Invariant of the above Quantic is equal to the determinant of \(mn - nm\), and that when it vanishes \(1, m, n, mn\), as also \(1, n, m, nm\) are linearly related—or, as I express it, this Invariant is the Involutant of the system \(m, n\) or \(n, m\). When \(m, n\) are of higher than the second order, the Involutant of \(m, n\), say \(I\), is that function whose vanishing implies that the 9 matrices \((1, m, m^21, n, n^2)\) are linearly related, and the Involutant of \(n, m\), say \(J\), that function whose vanishing implies that the 9 quantities \((1, n, n^21, m, m^2)\) are so related \((I, J\) being two distinct functions), and so for matrices of any order higher than the second.
By virtue of a general theorem for any two matrices \( m, n \) of the second order, the following identities are satisfied:
\[
\begin{align*}
m^2 - 2bm + d &= 0, \\
mn + nm - 2bm - 2cm + 2e &= 0, \\
n^2 - 2cn + f &= 0.
\end{align*}
\]

If then \( mn + nm = 0 \), since \( m \) and \( n \) cannot be functions of one another (for then \( mn = nm \)), the second equation shows that \( b = 0, \ c = 0, \ e = 0 \), and conversely if \( b = 0, \ c = 0, \ e = 0 \), \( mn + nm = 0 \), and \( m^2 + d = 0, \ n^2 + f = 0 \), where, if we please, we may make \( d = 1, \ f = 1 \).

(2) Let \( m, n \) be matrices of the third order, and write as before,
\[
\text{Det. } (x + ym + zn) = x^3 + 3bx^2y + 3cx^2z + 3dxyz + 6exyz + 3fzx^2 + gy^3 + 3hy^2z + 3kyz^2 + lz^3.
\]

Then by virtue of the general theorem last referred to* there exist the identical equations
\[
\begin{align*}
m^2 &= 3bn^2 + 3dm = g = 0, \\
m^2n + nm + nm^2 &= 3b(mn + nm) - 3cm^2 + 3dn + 6em - 3h = 0, \\
mn^2 + n^2m - 3e(mn + nm) - 3bn^2 + 3fm + 6en - 3k = 0, \\
n^2 - 3cn^2 + 3fn - l &= 0.
\end{align*}
\]

Let now \( mn = \rho mn \), where \( \rho \) is either imaginary cube root of unity, then

(1) \( m^2 + mn + nm^2 = 0 \) and (2) \( mn^2 + nm + n^2m = 0 \);

for greater simplicity suppose also that \( m^2 = n^2 = 1 \), where \( 1 \) means the matrix
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

From the 1st and 2nd of the four identical equations combined it may be proved that \( b = 0, \ d = 0 \); I do not produce the proof here because to make it rigorous, the theory of Nullity would have to be gone into which would occupy too much space; and in like manner from the 3rd and 4th it may be shown that \( c = 0, \ f = 0 \). Hence returning to the two middle equations it follows that \( e = 0, \ h = 0, \ k = 0 \), and from the two extremes that \( g = 1, \ l = 1 \).

If then \( mn = \rho mn, \ m^2 = 1, \) and \( n^2 = 1 \), it is necessary that
\[
b = 0, \ c = 0, \ d = 0, \ e = 0, \ f = 0, \ g = 1, \ h = 0, \ k = 0, \ l = 1.
\]

But these equations although necessary are manifestly insufficient; for they lead to the equations \( m^2 - 1 = 0, \ n^2 - 1 = 0 \), and

(1) \( m^2n + nm + nm^2 = 0 \); (2) \( mn^2 + nm + n^2m = 0 \),

[* By Cayley's theorem, if in \( \text{Det. } (x + ym + zn) \) we replace \( x \) by \( -ym - zn \), the result vanishes identically in regard to \( y \) and \( z \).]

† Except when \( m, n \) are functions of one another, so that \( mn \) and \( nm \) are identical and consequently are each of them zero.
but not necessarily to \( mn = pmn \). In fact the supposed equations between
\( m \) and \( n \) involve as a consequence the equation \( (mn)^2 = 1 \). Now the general
identical equation to \((mn)\) is

\[
(mn)^2 - 3B(mn)^2 + 3D(mn) - F = 0,
\]

where \( B \) is the sum of each term in \( m \) by its altruistic opposite in \( n \)
\( = 3bc - 2g = 0 \), \( F = gl = 1 \), and \( D \) is the sum of each first minor in \( m \)
by its altruistic opposite in \( n \) which sum does not necessarily vanish when
\( b, c, d, e, f, h, k \), all vanish. Hence there is a 10th condition necessary not
involved in the other 9, namely, \( D = 0 \). These 10 conditions I shall show
are sufficient as well as necessary. For when they are satisfied since
\( (mn)^2 = 1 \), \( mn . mn = n^2m^2 \).

Hence from (1)
\[
m^2n^2 + n^2m^2 + nm^2n = 0,
\]
and from (2)
\[
m^2n^2 + n^2m^2 + mn^2m = 0.
\]
Hence \( mn . mn = mn . nm \), and consequently \( mn \) is a function of \( mn \) [cf. p. 149,
below]. Hence we may write
\[
mn = A + Bmn + C(mn)^2.
\]

But the latent roots of \( mn \) and \( nm \) (which are always identical) are
\( 1, \rho, \rho^2 \), hence
\[
A + B + C, \quad A + B\rho + C\rho^2, \quad A + B\rho^2 + C\rho,
\]
must be equal to \( 1, \rho, \rho^2 \), each to each taken in some one of the 6 orders in
which these quantities can be written.

Solving these 6 systems of linear equations there results:
\[
A = 0, \quad B = 0, \quad C = 1, \rho \text{ or } \rho^2
\]
or
\[
A = 0, \quad B = 1, \rho \text{ or } \rho^2, \quad C = 0.
\]
Hence \( nm = \theta mn \), or \( \theta (mn)^2 \) where \( \theta = 1, \rho, \rho^2 \).

If
\[
nm = \theta (mn)^2, \quad nmnm = \theta (mn)^2 = \theta.
\]
Hence
\[
m^2 = \theta n^2, \quad \theta n^2 = \theta^2 n;
\]
and
\[
m^2n + nmnm + nm^2 = 3\theta m^2 = 3\theta m = 0,
\]
so that \( m = 0 \), and \( m^2 = 0 = 1 \); and again if \( nm = mn \),
\[
m^2n + nmnm + nm^2 = 2m^2n + nmnm = 3m^2n = 0,
\]

* This equation is independent of the equation \( (mn)^2 = 1 \); for
\[
m^2n - nm^2m = (m^2n + nmnm + nm^2) n - m (mn^2 + nmnm + n^2m) = 0
\]
by virtue of equations (1) and (2) above: accordingly these equations taken alone imply the equations
\[
nm = A + B_1 mn + C (mn)^2, \quad mn = - A + B_2 mn - C (mn)^2
\]
where \( B_1, B_2 \) are the roots of \( B^2 + B + 1 - \frac{AC}{2} = 0 \); \( A, C \) being arbitrary and independent except
that each vanishes when and only when the cube of \( mn \) and (as a consequence) of \( nm \), is a scalar
matrix. [See below, p. 154. Footnote [†].]

† By virtue of the general theorem that the latent roots of any function of a matrix are the
like functions of the latent roots of the original matrix.
so that \( m^2 n = 0, n = 0, \) and \( n^2 = 0 = 1 \) as before, where it should be noticed
\[
\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}
\]
that \( 0 = 1 \) means that \( 0 \ 0 \ 0 \) is identical with \( 0 \ 1 \ 0 \).

Hence the only available hypothesis remaining is the equation \( nm = v . mn \), where \( v \) is one of the imaginary cube-roots of unity as was to be proved.

(3) It remains to say a few words on the general equation \( nm = kmn \), where \( m, n \) are matrices of any order \( \omega \). To avoid prolixity I shall confine my remarks to the general case, which is, that where the determinants (or as I am used to say the contents) of \( m \) and \( n \) are each of them finite; with this restriction, the proposed equation is impossible for general values of \( k \) as will be at once obvious from the fact that the totalities of the latent roots of \( mn \) and of \( nm \) are always identical, but the individual latent roots are by virtue of the proposed equation in the ratio to one another of \( 1 : k \), which, since by hypothesis no root is zero, is only possible when \( k^\omega = 1 \).

When the above equation is satisfied the \( \omega^2 \) equations arising from the identification of \( nm \) with \( kmn \) cease to be incompatible and (as is necessary or at all events usual in such a contingency) become mutually involved. Thus, for example, when \( \omega = 1 \) and \( k = 1 \), the number of independent equations is \( 0 \), that is, \( 1 - 1 \), when \( \omega = 2 \) and \( k = -1 \) the number is \( 3 \), that is, \( 4 - 1 \), when \( \omega = 3 \) and \( k = \rho \) or \( \rho^2 \) the number is \( 8 \), that is, \( 9 - 1 \); it is fair therefore to presume (although the assertion requires proof) that for any value of \( \omega \) when \( k \) is a primitive \( \omega \)th root of unity the number of conditions to be satisfied when \( nm = kmn \) is \( \omega^2 - 1 \). Of these the condition that the content of \( x + my + nz \) shall be of the form \( x^\omega + cy^\omega + cz^\omega \) will supply
\[
\frac{(\omega + 1)(\omega + 2)}{2} - 3, \quad \text{that is,} \quad \frac{\omega^2 + 3\omega}{2} - 2,
\]
and there will therefore be
\[
\frac{\omega^2 - 3\omega}{2} + 1 \quad \text{or} \quad \frac{(\omega - 1)(\omega - 2)}{2}
\]
to be supplied from some other source.

When \( k \) is a non-primitive \( \omega \)th root of unity, the number of equations of condition is no longer the same. Thus when \( k = 1 \) we know that \( n \) may be of the form
\[
A + Bm + Cm^2 + \ldots + Lm^{\omega-1},
\]
where \( A, B, \ldots, L \), and all the \( \omega^2 \) terms in \( m \) are arbitrary, and consequently the number of conditions for that case is \( 2\omega^2 - (\omega^2 + \omega) \) or \( \omega^2 - \omega \). It seems then very probable that if \( k \) is a \( q \)th power of a primitive \( \omega \)th root of unity the number of conditions required to satisfy \( nm = kmn \) is \( \omega^2 - \delta \) where \( \delta \) is
the greatest common measure of \(q\) and \(\omega\): but, of course, this assertion awaits confirmation.

When \(\omega = 4\) besides the case of \(nm = mn\), that is, of \(n\) being a function of \(m\) of which the solution is known, there will be two other cases to be considered, namely, \(nm = -mn\) and \(nm = imn\): the former probably requiring 14 and the latter 15 conditions to be satisfied between the coefficients of \(m\), the coefficients of \(n\) and the two sets of coefficients combined.

It is worthy of notice that the conditions resulting from the content of \(x + my + nz\) becoming a sum of 3 powers are incompatible with the equation \(nm = vmn\) when \(v\) is other than a primitive \(\omega\)th root of unity (\(\omega\) being of course the order of \(m\) or \(n\)).

Thus suppose \(\omega = 4\); the conditions in question applied to the middle one of the 5 identical equations give
\[m^2n^2 + n^2m^2 + mn^2m + nm^2n + nmnm + nmnm = 0;\]
when \(nm = imn\) the left-hand side of this equation becomes
\[(1 + i^2 + i^2 + i + i^2)m^2n^2,\]
that is, is zero, but when \(nm = -mn\), the value is
\[(1 + 1 - 1 - 1 - 1)m^2n^2\]
which is not zero, and so in general. Thus the pure power form of the content of \(x + my + nz\) is a condition applicable to the case of \(\frac{nm}{mn}\) being a primitive root of unity and to no other.

The case of \(nm\) being a primitive root of ordinary unity is therefore the only one which it is most interesting to thrash out.

There are in this case, we have seen, \(\frac{1}{\omega} (\omega^2 + 3\omega - 4)\) simple conditions expressible by the vanishing of that number of coefficients in the content of \(x + my + nz\) and \(\frac{1}{\omega} (\omega - 1)(\omega - 2)\) supplemental ones. What are these last? I think their constitution may be guessed at with a high degree of probability. For revert to the case of \(\omega = 3\) in which there is one such found by equating to zero the second coefficient in the identical equation
\[(mn)^2 - 3B (mn)^2 + 3Dmn - G = 0.\]

Suppose now \((m^2n^2)^2 - 3B' (m^2n^2)^2 + 3D'm^2n^2 - G' = 0\)
is the identical equation to \(m^2n^2\). By virtue of the 8 conditions supposed to be satisfied we know that \(nm = pmn\) as well as \(m^2 = 1\), \(n^2 = 1\), and consequently that \((m^2n^2)^2 = 1\). Hence \(B' = 0\), \(D' = 0\), by virtue of the 7 parameters in the oft-quoted content and of \(D\) being all zero, and thus the evanescence of \(B'\) or \(D'\) imports no new condition.
Now suppose $\omega = 4$, and that
\[(mn)^4 - 4B(mn)^3 + 6D(mn)^2 - 4Gmn + M = 0,
(mn^2)^4 - 4B'(mn^2)^3 + 6D'(mn^2)^2 - 4G'mn^2 + M' = 0.
\]
Here we know that $B$ vanishes by virtue of $b$, $c$ and $e$ vanishing, but $D = 0$, $G = 0$, which must be satisfied if $nn = imn$, will be two new conditions not implied in those which precede. It seems then, although not certain, highly probable that $B' = 0$, $D' = 0$, will be implied in the satisfaction of the antecedent conditions but that $G' = 0$ will be an independent condition, so that $D = 0$, $G = 0$, $G' = 0$, will be the three supplemental conditions: and again when $\omega = 5$ forming the identical equations to $mn$, $m^2n^2$, $m^3n^3$, and using an analogous litteration to what precedes, the supplemental conditions will be
\[
D = 0, \quad G = 0, \quad M = 0,
G' = 0, \quad M' = 0,
M'' = 0,
\]
and so in general for any value of $\omega$.

The functions $D$, $G$, $M$, etc., above equated to zero are known from the following theorem of which the proof will be given in the forthcoming memoir*.

If
\[
(mn)^n + k_1(mn)^{n-1} + \ldots + k_i(mn)^{n-i} + \ldots = 0
\]
is the identical equation to $mn$, then $k_i$ is equal to the sum of the product of each minor of order $i$ in $m$ multiplied by its altruistic opposite in $n$.

The annexed example will serve to illustrate in the case of $\omega = 3$ that unless the supplemental condition is satisfied we cannot have $nm = \rho mn$.

Write
\[
m = 1 \quad 0 \quad 0, \quad n = 0 \quad c \quad k,
0 \quad \rho \quad 0, \quad k \quad 0 \quad cp,
0 \quad 0 \quad \rho^2, \quad cp^2 \quad k \quad 0,
\]
then the determinant to $x + my + nz$ will be easily found to be
\[
x^3 + y^3 + (\rho^2 + k^2)z^3;
\]
but $D$ becomes $-3\rho e k$, and does not vanish unless $c = 0$ or $k = 0$, and accordingly we find
\[
nm = 0 \quad \rho c \quad \rho^2 k, \quad nm = 0 \quad c \quad k,
k \quad 0 \quad c, \quad pk \quad 0 \quad \rho c,
\rho c \quad pk \quad 0, \quad \rho c \quad \rho^2 k \quad 0.
\]
When $k = 0$ $mn = \rho^2 mn$, when $c = 0$ $nm = \rho^4 mn$, but on no other supposition will $\frac{nm}{mn}$ be a primitive cube root of unity.

* This theorem furnishes as a Corollary the principle employed to prove the stability of the Solar System. (See Lond. and Edin. Phil. Mag., October, 1883.) [Above, p. 110.]
ADDENDUM.

Referring to the equation $MN = -NM$, and to the eight equations expressing $M$ and $N$ in terms of the combinations of the powers of $m$ with those of $n$, in which it is to be understood that $M$ and $N$ are non-vacuous, we know that the sums of the latent roots of $M$ and of $N$ must each vanish and consequently, as may be proved, that $a = 0$, $a' = 0$, leaving 8 or 6 conditions to be satisfied. If we further stipulate that $M^2 = 1$, $N^2 = 1$, there will be 8 relations connecting the coefficients $b, c, ..., k$ and $b', c', ..., k'$, so that the 64 coefficients in the 8 equations connecting $M, M^2; N, N^2; MN, M^2N^2; M^2N, MN^2$, or say rather $M, M^2; N, N^2; \rho MN, \rho M^2N^2; \rho M^2N, \rho MN^2$, with like combinations or multiples of combinations of powers of $m, n$, will be connected together by 56 equations; the coefficients in the expression for any one of the above 8 terms may then be arranged in pairs $f_i, f'_i; g_i, g'_i; h_i, h'_i; k_i, k'_i$; and in the expression for its fellow by $F_i, F'_i; G_i, G'_i; H_i, H'_i; K_i, K'_i$; so that the Matrix is resolved as it were into 4 sets of paired columns and 4 sets of paired lines; the 4 different sets of paired lines being found by writing successively $i = 1, 2, 3, 4$.

It is then easy to see that there will be 4 equations of the form

$$\Sigma (f_a G_{a'} + f_{a'} G_a) = 1,$$

and 6 quaternary groups (that is, 24 equations) of the form

$$\Sigma (f_a G_{a'} + f_{a' a}) = 0,$$

with liberty to change $f$ into $F$ or $G$ into $g$ or each into each: together then the above are 28 of the 56 conditions required. But inasmuch as the 8 $[m, n]$ arguments may be interchanged with the 8 $[M, N]$ ones, we may transform the above equations by substituting for each letter $f$ its conjugate $\frac{d \log \Delta}{df}$ (where $\Delta$ is the content of the Matrix) and thus obtain 28 others, giving in all (if the two sets as presumably is the case are independent) the required 56 conditions: the latter 28, however, may be replaced by others of much simpler form $\dagger$.

$\dagger$ It is easy to see that the sum of the latent roots of $M^2N^2$ must be zero for all values of $i, j$ so that it is a homogeneous linear function of the 8 quantities $m, m^2, ..., mn, m^2n^2$.

$\dagger$ I am still engaged in studying this matrix, which possesses remarkable properties. Is it orthogonal? I rather think not, but that it is allied to a system of 4 pairs of somethings drawn in four mutually perpendicular hyperplanes in space of 4 dimensions. In the general case of $MN = \rho NM$ where $\rho$ is a primitive $\omega$th root of unity, there will be an analogous matrix of the order $\omega - 1$ where each line and each column will consist of $\omega + 1$ groups of $\omega - 1$ associated terms.

The value of the cube of any one of the 8 matrices $M, M^2; ..., MN, M^3N^2$ may be expressed as follows: It is $P$ into ternary unity. Such a quantity may be termed by analogy a Scalar. To find $x_{ij}$ I imagine the 8 letters corresponding to $M^2N^2$ (but without powers of $p$ attached) to be set over 8 of the 9 points of inflexion to any cubic curve, the paired letters being made suitably
To me it seems that this vast new science of multiple quantity soars as high above ordinary or quaternion Algebra as the Mécanique Céleste above the "Dynamics of a Particle" or a pair of particles, (if a new Tait and Steele should arise to write on the Dynamics of such pair,) and is as well entitled to the name of Universal Algebra as the Algebra of the past to the name of Universal Arithmetic.

collinear with the missing 9th point. Then among themselves the 8 letters may be taken in 8 different ways to form collinear triads and the product of the letters in each triad may be called a collinear product; \( P_{ij} \) (which is identical with the Determinant to \( M_1N_3 \)) will be the sum of the cubes of the 8 letters less 3 times the sum of their 8 collinear products, and its 8 values will be analogous to the 8 values of the sum of 3 squares in the Quaternion Theory. Each of these 8 values is assumed equal to unity.

It may be noted that the product of four squares by four is representable rationally as a sum of four squares, so if we place (not now 8 specially related but) nine perfectly arbitrary letters over the nine points of inflexion of a cubic curve the sum of their 9 cubes less three times their 12 collinear products multiplied by a similar function of 9 other letters may be expressed by a similar function of 9 quantities lineal-linear functions of the two preceding sets of 9 terms.

By the 8 letters of any set as, for example, \( b, ..., h' \) being "specialized," I mean that they are subject to the condition \( bb' + dd' + ff' + hh' = 0 \). When this equation is satisfied, and not otherwise, \( M^3 \) will be a Scalar, and it must be satisfied when \( MN = pNM \).
ON INVOLUTANTS AND OTHER ALLIED SPECIES OF INVARIANTS TO MATRIX SYSTEMS.

[Johns Hopkins University Circulars, III. (1884), pp. 9—12, 34, 35.]

To make what follows intelligible I must premise the meaning and laws of vacuity and nullity.

A matrix is said to be vacuous when its content (the determinant of the matrix) is zero, but it may have various degrees of vacuity from 0 up to \( \omega \) the order of the matrix.

If from each term in the principal diagonal of a matrix \( \lambda \) be subtracted, the content of the resulting matrix is a function of degree \( \omega \) in \( \lambda \); the \( \omega \) values of \( \lambda \) which make this content vanish are called its latent roots, and if \( i \) of these roots are zero, the vacuity (treated as a number) is said to be \( i \). This comes to the same thing as saying that the vacuity is \( i \) when the determinant, and the sums of the determinants of the principal minors of the orders \( \omega - 1, \omega - 2, \ldots (\omega - i + 1) \) are each zero. A principal minor of course means one which is divided into 2 [equal] triangles by the principal diagonal of the parent matrix.

Again the nullity is said to be \( i \) when every minor of the order \( (\omega - i + 1) \), and consequently of each superior order, is zero. It follows therefore that it means the same thing to predicate a vacuity 1 and a nullity 1 of any matrix, but for any value of \( i \) greater than 1, a nullity \( i \) implies a vacuity \( i \) but not \emph{vice versa}; the vacuity may be \( i \), whilst the nullity may have any value from 1 up to \( i \) inclusive.

The law of nullity which I am about to enunciate is one of paramount importance in the theory of matrices*.

* The three cardinal laws or landmarks in the science of multiple quantity are (1) the law of \emph{nullity}, (2) the law of \emph{latency}, namely, that if \( \lambda_1, \lambda_2, \ldots \lambda_\omega \) are the latent roots of \( m \), then \( f\lambda_1, f\lambda_2, \ldots f\lambda_\omega \) are those of \( fm \), including as a consequence that

\[
f_{m} = 2\sqrt{\lambda_1 (m - \lambda_2) \ldots (m - \lambda_\omega)},
\]

and (3) the law of \emph{identity}, namely, that the powers and combinations of powers of two matrices \( m, n \) of the order \( \omega \) are connected together by \( (\omega + 1) \) equations whose coefficients are all included among the coefficients of the determinant to the Matrix

\[
x + ym + zn.
\]
The law is that the nullity of the product of two (and therefore of any number of) matrices cannot be less than the nullity of any factor nor greater than the sum of the nullities of the several factors which make up the product.

Suppose now that $\lambda_1, \lambda_2, \ldots \lambda_\omega$ are the latent roots of any matrix with unequal latent roots of the order $\omega$. It is obvious that any such term as $m - \lambda_1$ will have the nullity 1, for its latent roots will be $0, \lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \ldots \lambda_\omega - \lambda_1$, and consequently its vacuity is 1.

Moreover we know from Cayley’s famous identical equation that the nullity of the product of all the $\omega$ factors is $\omega$.

Hence it follows that if $M_i$ contains $i$, and $M_j$ the remaining $\omega - i$ of these factors (so that $i + j = \omega$), the nullity of $M_i$ must be exactly $i$ and of $M_j$ exactly $j$.

For the theorem above stated shows that $M_i$ cannot have a nullity greater than $i$, nor $M_j$ a nullity greater than $j$.

Hence if the nullity of the one were less than $i$ or of the other less than $j$, the nullity of $M_iM_j$ would be less than $i + j$, that is, less than $\omega$, whereas its nullity is $\omega$; hence the two nullities are respectively $i$ and $j$ as was to be shown.

Furthermore we know that the latent roots of $(m - \lambda_1)^i$ are $(\lambda_1 - \lambda_1)^i$; $(\lambda_2 - \lambda_1)^i$; \ldots $(\lambda_\omega - \lambda_1)^i$.

Hence if the latent roots of $m$ are all distinct, the nullity of $(m - \lambda_i)^i$ is unity and consequently by the same reasoning as that above employed it follows that the nullity of

$$(m - \lambda_1)^i, (m - \lambda_2)^i, \ldots (m - \lambda_i)^i$$

is exactly $i$.

I will now explain what is meant by the Involutant or Involutants of a system of two matrices of like order.

It will be convenient here to introduce the term "topical resultant" of a system of $\omega^2$ matrices each of order $\omega$.

We may denote any matrix say

$$
\begin{array}{cccc}
  a_{11} & a_{12} & \ldots & a_{1\omega} \\
  a_{21} & a_{22} & \ldots & a_{2\omega} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{\omega1} & a_{\omega2} & \ldots & a_{\omega\omega} \\
\end{array}
$$

by the linear form

$$
\begin{align*}
  a_{11} t_{11} + a_{12} t_{12} + \ldots + a_{1\omega} t_{1\omega} \\
  + a_{21} t_{21} + a_{22} t_{22} + \ldots + a_{2\omega} t_{2\omega} \\
  \vdots \vdots \ddots \vdots \\
  + a_{\omega1} t_{\omega1} + a_{\omega2} t_{\omega2} + \ldots + a_{\omega\omega} t_{\omega\omega}
\end{align*}
$$
where the \( t \) system is the same for all matrices of the order \( \omega \). If, then, we have \( \omega^2 \) such matrices, their topical resultant is the Resultant in the ordinary sense of the \( \omega^2 \) linear forms above written, proper to each of them respectively.

Suppose now that \( m, n \) are two independent matrices of the order \( \omega \), we may form \( \omega^2 \) matrices by taking each power of \( m \) from 0 to \( \omega - 1 \) as an antecedent factor, and can combine it with similar powers of \( n \) as a consequent factor, and in this way obtain \( \omega^2 \) matrices, of which the first will be the \( \omega \)-ary unity, that is, a matrix of the order \( \omega \) in which the principal diagonal terms are all units and the other terms all zero. The topical resultant of these \( \omega^2 \) matrices I shall for brevity denote as the Involutant to \( m, n \).

In like manner, inverting the position of the powers of \( m \) and of \( n \) so as to make the latter precede instead of following the former in the \( \omega^2 \) products above referred to, we shall obtain another topical resultant which may be termed the Involutant to \( n, m \).

The reason why I speak of these topical resultants as involutants to \( m, n \) or \( n, m \) is the following:

In general if \( m, n \) are two independent matrices, any other matrix \( p \), by means of solving \( \omega^2 \) linear equations, may obviously be expressed as a linear function of the \( \omega^2 \) products

\[
(1, m, m^2, \ldots, m^{\omega-1})(1, n, n^2, \ldots, n^{\omega-1}).
\]

There are, however, exceptions to this fact.

The most obvious exception is that which takes place when \( n \) is a function of \( m \); for then any \( \omega \) of the \( \omega^2 \) products will be linearly related, and there will be substantially only \( \omega \) disposable quantities to solve \( \omega^2 \) equations.

Another exception is when the \( m, n \) Involutant, that is, the topical resultant of the \( \omega^2 \) matrices, is zero; in which case the general values of the \( \omega^2 \) disposable quantities each becomes infinite. So that \( m, n \) may be said to be in a kind of mutual involution with one another. So, again, \( p \) may in general be expressed as a linear function of the \( \omega^2 \) matrices

\[
(1, n, n^2, \ldots, n^{\omega-1})(1, m, m^2, \ldots, m^{\omega-1}),
\]

but when the \( n, m \) Involutant vanishes this is no longer possible.

When \( \omega = 2 \) the two involutants, considered as definite determinants, are absolutely equal in magnitude and in Algebraical sign, but when \( \omega \) exceeds 2 this is no longer the case; the two Involutants are then entirely distinct functions of the elements of \( m \) and \( n \).
Thus to take a simple example: if \( m = 0 \), \( \rho = 0 \) and \( n = k \), \( 0^2 \) it will
be found by direct calculation of two topical resultants of the 9th order,
that the two involutants will be

\[
81(\rho - \rho^2)(k^3 - \rho^3) \quad \text{and} \quad 81(\rho^3 - \rho)(k^3 - \rho^3)^3
\]

respectively. The reason why the two involutants coincide in the case of
\( \omega = 2 \) is not far to seek. It depends upon the fact of the existence of the
mixed identical equation

\[
mn + nm - 2bn - 2cm + 2s = 0;
\]

from which it is obvious that the topical resultant of 1, \( m, n, mn \) is the
negative of that of 1, \( m, n, nm \) or identical with that of 1, \( n, m, nm \).

By direct calculation it will be found that the Involutant \( m, n, o r n, m, \)
where \( m = \frac{f}{k} \quad g \quad \frac{f'}{k'} \quad \frac{g'}{k'} \;
\]

\[
- (gh' - g'k) + [(f - k)h' - (f' - k')g'][(f - k)h' - (f' - k')h],
\]

which is the same thing as the content of the matrix \( (mn - nm) \). It may
also be shown \( \text{à priori} \) or by direct comparison to be identical (to a numerical
factor \( \text{près} \)) with the Discriminant of the Determinant to the matrix
\( (x + ym + zn) \) which is a ternary quantic of the second order. Its actual
value is 4 times that discriminant.

Let us consider the analogous case of Mechanical Involution of lines in
a plane or in space. There are two questions to be solved. The one is to
find the condition that the Involution may exist, that is, that a set of
equilibrating forces admit of being found to act along the lines; the second,
to determine the relative magnitudes of the forces when the involution
exists, and this is the simpler question of the two.

In like manner we may consider two questions in the case of \( m, n \) being
in either of the two kinds of involution; the one being to find what the
condition is of such involution existing, the other what are the coefficients of
the \( \omega^a \) coefficients in the equation which connects the \( \omega^a \) products, when the
involution exists.

This latter part of the question (surprising as the assertion may appear
and is) admits of a very simple and absolutely general direct and almost
instantaneous solution by means of the Law of Nullity, above referred to,
as I will proceed to show.

The determination of the Involutants, or at all events of their product,
will then be seen to follow as an immediate consequence from this prior
determination of the form of the equations which express the involutions of
the two kinds respectively.
But first it may be well to explain why and in what sense I refer in the title to Involutants as belonging to a class of invariants. I say, then, that universally involutants are invariants in this sense, that if for $m$ and for $n$, any function of $m$, or any function of $n$ be substituted, the ratio of the two Involutants, say $I$ and $J$, remains unaltered. By virtue of the Identical Equation $(m)^i$ will be of the form of

$$A_i + B_i + C_i m^2 + \ldots + L_i m^{i-1}$$

and as a consequence it is easy to see that when $m^i$ is substituted for $m$, $I$ and $J$ will become respectively $P_i J$, $P J$, where $P$ is the $\omega$th power of the determinant to the matrix formed by writing under one another the $(\omega - 1)$ lines of terms, of which the line $B_i$, $C_i$, $\ldots$; $L_i$ is the general expression.

Moreover, in the particular case where $\omega = 2$ and $I = J^*$, besides being an Invariant in this modified sense, $I$ will be an invariant in a sense including but transcending the more ordinary conception of an Invariant; for if when, for $m$ and $n$, $f(m, n)$ and $\phi(m, n)$ are substituted, $I$ becomes $I'$, then $I'$ will contain $I$ as a factor; this is a consequence of the fact that when $m$ and $n$ are in involution $f(m, n)$ and $\phi(m, n)$ will also be in involution, for in consequence of the identical equation

$$mn + nm - 2bm - 2cn + 2e = 0$$

$f$ and $\phi$ and $f\phi$ will each be reducible to the form

$$A + Bm + Cn + Dmn$$

and it is obvious from the ordinary theory of the determinants that the topological resultant of $1$, ($\text{meaning } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$), and three linear functions of $1$, $m$, $n$, $nm$, will contain as a factor the topological resultant of $1$, $m$, $n$, $mn$.

Nor must it be supposed that Involutants are the only species of invariants in the modified sense first described which appertain to the

* I for some time had imagined, and indeed thought I had proved, that the two involutants were always identical. When crossing the Atlantic last month on board the "Arizona," having hit upon a pair of matrices of the third order, for which the two topological resultants admitted of easy calculation, I found, to my surprise, that they were perfectly distinct. The cause of the failure of the supposed proof constitutes a paradox which will form the subject of a communication to a future meeting of the Johns Hopkins Mathematical Society.

I will here only premise that the seeming contradiction between the logical conclusion and the facts of the case takes its rise in a sort of mirage with which invariantists are familiar, namely: the apparent a priori establishment of algebraical forms as the result of perfectly valid processes, which forms have no more real existence in nature than the Coronus of the Sun under our Dr Hastings' scrutinizing gaze: the contradiction between the logical inference and the truth being accounted for by the circumstance that any such supposed form on actual performance of the operations indicated, turns out to be a congeries of terms, each affected with a null coefficient; we are thus taught the lesson that all a priori reasoning until submitted to the test of experience, is liable to be fallacious, and it is impossible to prove that a proof may not be erroneous by any other method than that of actual trial of the results which it is supposed to yield.