

bas de la série, et en me servant de  $Y$  ou  $Z$  pour signifier un symbole ou simple ou affecté de marques quelconques, j'emploie les notations

$$Y=0, \quad Y+Z=0, \quad Y>0, \quad Y+Z>0,$$

pour signifier que les  $Y$  manquent, que les  $Y$  et les  $Z$  manquent tous les deux, que les  $Y$  ne manquent pas, que les  $Y$  et les  $Z$  ne manquent pas tous les deux.

Je divise les  $B$  (d'un arrangement quelconque) en deux espèces, ' $B$ ' et ' $B'$ , dont ' $B$ ' représente un  $B$  appartenant à la série arithmétique (la plus grande qu'on puisse former) commençant avec le plus grand  $B$ , et ' $B'$  les autres  $B$  qui se trouvent dans l'arrangement.

Ainsi je divise les  $A$  en ' $A$ ' et en ' $A'$ ; ' $A$ ' signifie un  $A$  appartenant à la série arithmétique la plus grande qu'on puisse former, dont  $a$  est le terme minimum (de sorte que, si l'arrangement ne contient pas un  $a$ ,  $A$ , manque) et ' $A'$  signifie les autres  $A$  de l'arrangement.

Finalement un point au centre d'un symbole à droite ou à gauche signifiera ce symbole diminué ou augmenté respectivement de  $c$ .

On voit que dans cette notation les arrangements exceptionnels seront exprimés ainsi: ceux qui appartiennent à l'une des deux classes par les conditions ' $B - b = 0$ ' avec ' $A + C = 0$ ', et les autres par les conditions ' $B = 0$ ' avec ' $A + C = 0$ '.

Je divise les arrangements non exceptionnels en trois classes, dont les conditions seront respectivement les suivantes:

Première classe :

$$'B - b > 0 \text{ ou } ('B - b = 0 \text{ avec } C - c \leq 'B - b).$$

Deuxième classe :

$$'B - b = 0 \text{ avec } (C - c > 'B - b \text{ ou } C = 0, \text{ mais } A + C > 0),$$

$$\text{ou } B = 0 \text{ avec } (A = 0 \text{ ou } A - a \geq C).$$

Troisième classe :

$$B = 0 \text{ avec } A > 0 \text{ et } A - a < C \text{ et } 'A + C > 0.$$

Toutes les hypothèses possibles se trouvent comprises dans ces tableaux des arrangements exceptionnels et non exceptionnels.

A chacune des trois classes des derniers je vais assigner un opérateur qui peut être appliqué à chaque arrangement de cette classe et qui le transformera dans un autre arrangement appartenant à la même classe; cette disposition, appliquée deux fois successivement, reproduira l'arrangement sur lequel on opère, lequel ne changera pas la somme des éléments, mais changera chacun des deux caractères en sens opposé: c'est-à-dire que chacun des trois opérateurs que je vais définir, et que je nommerai  $\phi$ ,  $\psi$ ,  $\vartheta$ , doit

satisfaire à cinq conditions qu'on peut nommer *catholicité*, *homœogénèse*, *mutualité*, *inertie* et *éenantiotropie*.

1.  $\phi$  signifie que, si  $C = 0$  ou  $C - c > 'B - 'B$ , on doit former un nouveau  $C$ , en substituant, pour chaque ' $B$ ', ' $B$ ' (c'est-à-dire sa valeur diminuée de  $c$ ), et reconstituer l'inertie originale en ajoutant ensemble les  $c$  ainsi soustraits pour former un nouveau  $C$ , et que, dans le cas contraire,  $C$  doit être décomposé en simples  $c$ , dont on ajoutera un au premier ' $B$ ' (le  $B$  le plus grand), un au second ' $B$ ', etc., jusqu'à ce que tous les  $c$  dont on a à disposer soient épuisés.

2.  $\psi$  signifie que, si  $B > 0$  ou  $C = 0$ , ou  $C > 'B + A$ , on doit former un nouveau  $C$  en substituant à ' $B$ ' et  $A$  leur somme et que, dans le cas contraire,  $C$  doit être décomposé en ' $B$ ' et  $A$  si  $B > 0$  et en  $b$  et  $A$  si  $B = 0$ .

3.  $\wp$  signifie que, si  $C = 0$  ou  $C + \dot{A} = > A$ , il faut décomposer  $A$  en  $\dot{A}$ , et  $C$  ou en  $a$  et  $C$ , selon que  $A = 0 > 0$ , et que, dans le cas contraire, pour  $C$  et  $\dot{A}$ , il faut substituer leur somme. On sera satisfait en étudiant les conditions des trois classes que les  $\phi$ ,  $\psi$ ,  $\wp$  possèdent tous les trois cinq attributs voulus : la preuve en est facilitée en supposant que, dans chaque série des  $C$ , des  $B$  et des  $A$ , prise séparément, on suit un ordre régulier de grandeur dans l'arrangement de ces termes respectivement au multiple de  $c$  qui entre dans chacun d'eux.

Si l'on donne à  $a$  et à  $b$  des valeurs quantitatives (ce qui est toujours permis), et en particulier les valeurs 1 et 2 respectivement, on retombe sur le théorème d'Euler, mais (chose à noter) la correspondance donnée par le procédé général appliqué à ce cas ne sera nullement identique à la correspondance donnée par le procédé de Franklin. En effet, les arrangements exceptionnels ne seront pas les mêmes dans les deux méthodes : selon le procédé de Franklin, les arrangements non conjugables sont de la forme

$$i, i+1, \dots, 2i-1 \text{ ou } i+1, i+2, \dots, 2i,$$

tandis que la méthode actuelle donnera, comme non conjugués, les arrangements de la forme

$$1, 4, \dots, 3i-2 \text{ ou } 2, 5, \dots, 3i-1.$$

La méthode employée ici fournira elle-même toujours deux systèmes de correspondance absolument distincts, dont on obtient l'un, qui n'est pas exprimé, en échangeant entre eux les  $a$ ,  $A$  et les  $b$ ,  $B$ , car la méthode n'est pas symétrique dans son opération sur ces deux systèmes de lettres.

Ce cas est analogue à celui de la correspondance perspective entre deux triangles, laquelle peut être simple ou triple, comme je l'ai montré ailleurs. Jacobi, dans l'endroit cité, a fait la remarque que, pour  $a=1, b=2$ , en se servant du signe supérieur ( $\mp$ ) dans son théorème, on retombe sur le

théorème d'Euler et que, pour le cas de  $a = 1$ ,  $b = 1$ , en se servant du signe inférieur, sur un théorème donné (il y a longtemps par Gauss). On peut ajouter que, si avec cette supposition on se sert du signe supérieur, on obtient  $0 = 0$ , mais si l'on écrit  $a = 1 - \epsilon$ ,  $b = 1$ , en faisant  $\epsilon$  infinitésimal, on tombe (chose singulière) sur l'équation de Jacobi elle-même,

$$(1 - q)^3(1 - q^2)^3(1 - q^3)^3 + \dots = 1 - 3q + 5q^3 - 7q^6 + \dots$$

Puisque j'ai introduit le nom de l'auteur des *Fundamenta nova*, qu'on me permette la remarque que, dans les deux avant-dernières lignes de l'avant-dernière page de cet immortel Ouvrage, on trouve un théorème qui équivaut à l'équation

$$\frac{q}{1+q} - \frac{q^3}{1+q^3} + \frac{q^5}{1+q^5} - \dots = \frac{q}{1+q} - \frac{q^{1+2}}{1+q^2} + \frac{q^{1+2+3}}{1+q^3} - \dots ;$$

or, le premier cas du théorème intitulé: *Sur un théorème d'Euler*, contenu dans une Note précédente des *Comptes rendus*\* affirme que le nombre des séries arithmétiques avec lesquelles on peut exprimer  $n$  est égal au nombre des diviseurs impairs de  $n$ , laquelle considération mène immédiatement à une conséquence qu'on ne pourrait manquer d'observer (mais que M. Franklin, effectivement, a remarquée le premier) et qui s'exprime par l'équation

$$\frac{q}{1-q} + \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} + \dots = \frac{q}{1-q} + \frac{q^{1+2}}{1-q^2} + \frac{q^{1+2+3}}{1-q^3} + \dots ,$$

équation très ressemblante à l'autre et qui peut être combinée avec elle de manière à donner naissance à quatre autres équations de la même espèce.

On n'a pas besoin de dire que le théorème qui constitue la matière principale de cette Note, en faisant  $a = 1$  et en considérant  $b$  comme une quantité arbitraire, contient ou au moins conduit immédiatement au développement de  $\Theta_1 x$  dont Jacobi l'a traité comme conséquence.

[\* p. 95 above. Cf. p. 25 above.]

## 9.

### ON THE NUMBER OF FRACTIONS CONTAINED IN ANY “FAREY SERIES” OF WHICH THE LIMITING NUMBER IS GIVEN.

[*Philosophical Magazine*, xv. (1883), pp. 251—257; xvi. (1883),  
pp. 230—233.]

A *Farey series* (“suite de Farey”) is a system of all the unequal vulgar fractions arranged in order of magnitude, the numerator and denominator of which do not exceed a given number.

The first scientific notice of these series appeared in the *Philosophical Magazine*, Vol. XLVII. (1816), pp. 385, 386. In 1879 Mr Glaisher published in the *Philosophical Magazine* (pp. 321—336) a paper on the same subject containing a proof of their known properties, an important extension of the subject to series in which the numerators and denominators are subject to distinct limits, and a bibliography of Mr Goodwyn’s tables of such series. Finally, in 1881 Sir George Airy contributed a paper also to the *Philosophical Magazine* of that year, in which he refers to a table calculated by him “some years ago,” and printed in the Selected Papers of the *Transactions* of the Institution of Civil Engineers, which is in fact a Farey table with the logarithms of the fractions appended to each of them. Previous tables had only given the decimal values of such fractions. The drift of this paper is to point out a caution which it is necessary to observe in the use of such tables, and which limits their practical utility: this arises from the fact of the differences receiving a very large augmentation in the immediate neighbourhood of the fractions which are a small aliquot part of unity—a fact which may be inferred *a priori* from the well-known law discovered by Farey applicable to those differences, but to which the author of the paper makes no allusion.

In addition to the tables of Farey series by Goodwyn, Wucherer, an anonymous author mentioned in the Babbage Catalogue, and Gauss, referred to by Mr Glaisher in his Report to the Bradford Meeting of the British Association (1873), may be mentioned one contained in Herzer’s *Tabellen*

(Basle, 1864) with the limit 57, and another in Hrabak's *Tabellen-Werk* (Leipsic, 1876), in which the limit is taken at 50.

The writers on the theory are:—Cauchy (as mentioned by Mr Glaisher), who inserted a communication relating to it in the *Bulletin des Sciences par la Société Philomathique de Paris*, republished in his *Exercices de Mathématiques*; Mr Glaisher himself (*loc. cit.*); M. Halphen, in a recent volume of the Proceedings of the Mathematical Society of France; and M. Lucas, in the next following volume of the same collection. I am indebted to my friend and associate Dr Story for these later references.

For theoretical purposes it is desirable to count  $\frac{1}{j}$  as one of the fractions in a Farey series. The number of such fractions for the limit  $j$  then becomes identical with the sum of the *totients* of all the natural numbers up to  $j$  inclusive—a totient to  $x$  (which I denote by  $\tau x$ ) meaning the number of numbers less than  $x$  and prime to it. Such sum, that is,  $\sum_{x=1}^{x=j} \tau x$ , I denote by  $T_j$ . My attention was called to the subject by this number  $T_j$  expressing the number of terms in a function whose residue (in Cauchy's sense) is the generating function to any given simple denumerant (see *American Journal of Mathematics*, [Vol. III. of this Reprint, p. 605]); and I became curious to know something about the value of  $T_j$ . I had no difficulty in finding a functional equation which serves to determine its limits (see *Johns Hopkins University Circular*, Jan. and Feb. 1883\*). The most simple form of that equation (omitted to be given in the *Circular*) is

$$T_j + T \frac{j}{2} + T \frac{j}{3} + T \frac{j}{4} + T \frac{j}{5} + \dots = \frac{j^3 + j}{2},$$

(where, when  $x$  is a fraction,  $Tx$  is to be understood to mean  $T_j$ ,  $j$  being the integer next below  $x$ ); and from this it is not difficult to deduce by strict demonstration that  $T_j/j^2$ , when  $j$  increases indefinitely, approximates indefinitely near to  $3/\pi^2$ .

I have subsequently found that if  $ux$  be used to denote the sum of all the numbers inferior and prime to  $x$ , and  $U_j = \sum_{x=j}^{x=1} ux$ , then †

$$U_j + 2U \frac{j}{2} + 3U \frac{j}{3} + 4U \frac{j}{4} + \dots = \frac{j(j+1)(j+2)}{3}$$

(where  $Ux$ , when  $x$  is a fraction, means the  $U$  of the integer next inferior to  $x$ ). From this equation it is also possible to prove that  $U_j/j^3$ , when  $j$  becomes indefinitely great, approximates to  $1/\pi^2$ .  $U_j$ , it may be well to notice, is the sum of all the numerators of the fractions in a Farey series whose limit is  $j$ , just as  $T_j$  is the number of these fractions.

In the annexed Table the value of  $\tau x$  (the totient), of  $Tx$  (the sum-totient), and of  $3/\pi^2 \cdot x^2$  is calculated for all the values of  $x$  from 1 to 1000; and the

[\* See pp. 84, 89 above.]

[† The right side should be  $\frac{1}{12}j(j+1)(2j+1)$ .]

remarkable fact is brought to light that  $Tx$  is always greater than  $3/\pi^2 \cdot x^2$  (the number opposite to it), and less than  $3/\pi^2 \cdot (x+1)^2$ , the number which comes after the following one in the same table.

I have calculated in my head the first few values of  $Ux$ , and find (if I have made no mistake) that it obeys an analogous law, namely is always intermediate between  $1/\pi^2 \cdot x^3$  and  $1/\pi^2 \cdot (x+1)^3$ .

It may also be noticed that when  $n$  is a prime number,  $Tn$  is always nearer, and usually very much nearer, to the superior than to the inferior limit—as might have been anticipated from the circumstance that, when this is the case, in passing from  $n-1$  to  $n$  the  $T$  receives an augmentation of  $n-1$ , whereas its average augmentation is only  $\frac{3}{\pi^2}(2n-1)$ .

In like manner and for a similar reason, when  $n$  contains several small factors  $Tn$  is nearer to the inferior than to the superior limit. For instance, when  $n=210$ ,  $Tn=13414$  and  $3/\pi^2 \cdot n^2=13404.79$ .

TABLE of Totients, of Sum-totients, and of  $3/\pi^2$  into the Squares of all the Numbers from 1 to 1000 inclusive.

$$\left[ \frac{3}{\pi^2} = .30396355 \right].$$

$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$
1	1	1	.30	27	18	230	221.59	53	52	882	853.83
2	1	2	1.22	28	12	242	238.31	54	18	900	886.36
3	2	4	2.74	29	28	270	255.63	55	40	940	919.49
4	2	6	4.86	30	8	278	273.56	56	24	964	953.23
5	4	10	7.60	31	30	308	292.11	57	36	1000	987.58
6	2	12	10.94	32	16	324	311.26	58	28	1028	1022.54
7	6	18	14.90	33	20	344	331.01	59	58	1086	1058.10
8	4	22	19.46	34	16	360	351.38	60	16	1102	1094.27
9	6	28	24.62	35	24	384	372.35	61	60	1162	1131.05
10	4	32	30.40	36	12	396	393.93	62	30	1192	1168.44
11	10	42	36.78	37	36	432	416.12	63	36	1228	1206.43
12	4	46	43.77	38	18	450	438.92	64	32	1260	1245.03
13	12	58	51.37	39	24	474	462.32	65	48	1308	1284.25
14	6	64	59.58	40	16	490	486.34	66	20	1328	1324.07
15	8	72	68.39	41	40	530	510.96	67	66	1394	1364.49
16	8	80	77.81	42	12	542	536.19	68	32	1426	1405.53
17	16	96	87.84	43	42	584	562.02	69	44	1470	1447.17
18	6	102	98.48	44	20	604	588.47	70	24	1494	1489.42
19	18	120	109.73	45	24	628	615.52	71	70	1564	1532.28
20	8	128	121.58	46	22	650	643.19	72	24	1588	1575.75
21	12	140	134.05	47	46	696	671.45	73	72	1660	1619.82
22	10	150	147.12	48	16	712	700.33	74	36	1696	1664.51
23	22	172	160.79	49	42	754	729.82	75	40	1736	1709.80
24	8	180	175.08	50	20	774	759.91	76	36	1772	1755.69
25	20	200	189.98	51	32	806	790.61	77	60	1832	1802.20
26	12	212	205.48	52	24	830	821.92	78	24	1856	1849.31

Remarks: In April, 1970, I checked this table as follows: Values of  $T(n) = \phi(n)$  = Euler's function were checked against those given in Number-Divisor Tables (British Assoc. Math. Tables, Vol. VIII, 1940), the only error found being at  $n=688$ , where  $\phi(n)=336$ , not 536. (This was merely a misprint and did not affect Sylvester's calculation of  $T(n) = \Phi(n) = \sum_{m=1}^n \phi(m)$ .) The values of  $\Phi(n)$  were obtained on a desk calculator and found to be correct for  $n=1, 2, \dots, 1000$ . Finally, the desk calculator was used to verify that  $\Phi(n) > (3/\pi^2)n^2$  for  $n=1, 2, \dots, 1000$  except for  $n=820$ , where  $\Phi(n)=204376 < (3/\pi^2)n^2 = 204385.09$ , as in the present table. (Strangely, Sylvester failed to notice this!) The fact that  $\Phi(n) > (3/\pi^2)n^2$  for  $1 \leq n \leq 1000$  except for  $n=820$  was (over)

TABLE (continued).

$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$
79	78	1934	1897·04	134	66	5498	5457·97	189	108	10904	10857·88
80	32	1966	1945·37	135	72	5570	5539·74	190	72	10976	10973·09
81	54	2020	1994·31	136	64	5634	5622·11	191	190	11166	11088·90
82	40	2060	2043·85	137	136	5770	5705·09	192	64	11230	11205·31
83	82	2142	2094·01	138	44	5814	5788·68	193	192	11422	11322·34
84	24	2166	2144·77	139	138	5952	5872·88	194	96	11518	11439·97
85	64	2230	2196·14	140	48	6000	5957·69	195	96	11614	11558·21
86	42	2272	2248·12	141	92	6092	6043·10	196	84	11698	11677·06
87	56	2328	2300·70	142	70	6162	6129·12	197	196	11894	11796·52
88	40	2368	2353·90	143	120	6282	6215·75	198	60	11954	11916·59
89	88	2456	2407·70	144	48	6330	6302·99	199	198	12152	12037·26
90	24	2480	2462·10	145	112	6442	6390·83	200	80	12232	12158·54
91	72	2552	2517·12	146	72	6514	6479·29	201	132	12364	12280·43
92	44	2596	2572·75	147	84	6598	6568·35	202	100	12464	12402·93
93	60	2656	2628·98	148	72	6670	6658·02	203	168	12632	12526·03
94	46	2702	2685·82	149	148	6818	6748·29	204	64	12696	12649·75
95	72	2774	2743·27	150	40	6858	6839·18	205	160	12856	12774·07
96	32	2806	2801·33	151	150	7008	6930·67	206	102	12958	12899·00
97	96	2902	2860·00	152	72	7080	7022·77	207	132	13090	13024·54
98	42	2944	2919·27	153	96	7176	7115·48	208	96	13186	13150·68
99	60	3004	2979·15	154	60	7236	7208·80	209	180	13366	13277·43
100	40	3044	3039·64	155	120	7356	7302·72	210	48	13414	13404·79
101	100	3144	3100·73	156	48	7404	7397·26	211	210	13624	13532·76
102	32	3176	3162·44	157	156	7560	7492·40	212	104	13728	13661·34
103	102	3278	3224·75	158	78	7638	7588·15	213	140	13868	13790·52
104	48	3326	3287·67	159	104	7742	7684·51	214	106	13974	13920·32
105	48	3374	3351·20	160	64	7806	7781·47	215	168	14142	14050·72
106	52	3426	3415·34	161	132	7938	7879·04	216	72	14214	14181·73
107	106	3532	3480·08	162	54	7992	7977·22	217	180	14394	14313·34
108	36	3568	3545·44	163	162	8154	8076·01	218	108	14502	14445·57
109	108	3676	3611·40	164	80	8234	8175·41	219	144	14646	14578·40
110	40	3716	3677·96	165	80	8314	8275·41	220	80	14726	14711·84
111	72	3788	3745·14	166	82	8396	8376·02	221	192	14918	14845·89
112	48	3836	3812·92	167	166	8562	8477·24	222	72	14990	14980·54
113	112	3948	3881·31	168	48	8610	8579·07	223	222	15212	15115·81
114	36	3984	3950·31	169	156	8766	8681·50	224	96	15308	15251·68
115	88	4072	4019·92	170	64	8830	8784·55	225	120	15428	15388·16
116	56	4128	4090·14	171	108	8938	8888·20	226	112	15540	15525·25
117	72	4200	4160·96	172	84	9022	8992·46	227	226	15766	15662·94
118	58	4258	4232·39	173	172	9194	9097·33	228	72	15838	15801·24
119	96	4354	4304·43	174	56	9250	9202·80	229	228	16066	15940·15
120	32	4386	4377·08	175	120	9370	9308·88	230	88	16154	16079·67
121	110	4496	4450·33	176	80	9450	9415·57	231	120	16274	16219·80
122	60	4556	4524·19	177	116	9566	9522·87	232	112	16386	16360·53
123	80	4636	4598·66	178	88	9654	9630·78	233	232	16618	16501·87
124	60	4696	4673·74	179	178	9832	9739·29	234	72	16690	16643·82
125	100	4796	4794·43	180	48	9880	9848·42	235	184	16874	16786·38
126	36	4832	4825·72	181	180	10060	9958·15	236	116	16990	16929·55
127	126	4958	4902·63	182	72	10132	10068·49	237	156	17146	17073·32
128	64	5022	4980·14	183	120	10252	10179·44	238	96	17242	17217·70
129	84	5106	5058·26	184	88	10340	10290·99	239	238	17480	17362·70
130	48	5154	5136·98	185	144	10484	10403·15	240	64	17544	17508·30
131	130	5284	5216·32	186	60	10544	10515·92	241	240	17784	17654·51
132	40	5324	5296·26	187	160	10704	10629·30	242	110	17894	17801·32
133	108	5432	5376·81	188	92	10796	10743·29	243	162	18056	17948·74

4749.43

verified independently by M.L.N. Sarma (On the error term in a certain sum, Proc. Indian Acad. Sci. Sect. A 3(1936), 338), who was apparently unaware of Sylvester's work. A search of the journal Mathematics of Computation (Vols. 1-23(1969)) uncovered no mention of errors in Sylvester's table. It did reveal the existence of two unpublished tables of  $\Phi(n)$  (see Math. Comput. 4(1950), 29-30 and 162). R.A. MacLeod (J. London Math. Soc. 42(1967), 652-660) claimed to have proved that  $\Phi(n)/n^2 - 3/\pi^2$  achieves its minimum over  $n=1, 2, 3, \dots$  at  $n=1276$  and that this minimum is about  $-0.00002466$ . (His statement of the main result has a misprint or error in it.) Incidentally, I did not check carefully Sylvester's values

TABLE (continued).

$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$
244	120	18176	18096·77	299	264	27318	27174·65	354	116	38174	38091·50
245	168	18344	18245·41	300	80	27398	27356·72	355	280	38454	38307·01
246	80	18424	18394·66	301	252	27650	27539·40	356	176	38630	38523·12
247	216	18640	18544·51	302	150	27800	27722·69	357	192	38822	38739·85
248	120	18760	18694·97	303	200	28000	27906·59	358	178	39000	38957·18
249	164	18924	18846·04	304	144	28144	28091·10	359	358	39358	39175·13
250	100	19024	18997·72	305	240	28384	28276·21	360	96	39454	39393·68
251	250	19274	19150·01	306	96	28480	28461·93	361	342	39796	39612·83
252	72	19346	19302·90	307	306	28786	28648·26	362	180	39976	39832·60
253	220	19566	19456·40	308	120	28906	28835·20	363	220	40196	40052·97
254	126	19692	19610·51	309	204	29110	29022·75	364	144	40340	40273·95
255	128	19820	19765·23	310	120	29230	29210·90	365	288	40628	40495·54
256	128	19948	19920·56	311	310	29540	29399·66	366	120	40748	40717·74
257	256	20204	20076·49	312	96	29636	29589·03	367	366	41114	40940·55
258	84	20288	20233·03	313	312	29948	29779·01	368	176	41290	41163·96
259	216	20504	20390·18	314	156	30104	29969·59	369	240	41530	41387·98
260	96	20600	20547·94	315	144	30248	30160·79	370	144	41674	41612·61
261	168	20768	20706·30	316	156	30404	30352·59	371	312	41986	41837·85
262	130	20898	20865·28	317	316	30720	30545·00	372	120	42106	42063·69
263	262	21160	21024·86	318	104	30824	30738·01	373	372	42478	42290·15
264	80	21240	21185·05	319	280	31104	30931·64	374	160	42638	42517·21
265	208	21448	21345·84	320	128	31232	31125·87	375	200	42838	42744·87
266	108	21556	21507·25	321	212	31444	31320·71	376	184	43022	42973·15
267	176	21732	21669·26	322	132	31576	31516·16	377	336	43358	43202·04
268	132	21864	21831·88	323	288	31864	31712·22	378	108	43466	43431·53
269	268	22132	21995·11	324	108	31972	31908·88	379	378	43844	43661·63
270	72	22204	22158·95	325	240	32212	32106·15	380	144	43988	43892·34
271	270	22474	22323·39	326	162	32374	32304·03	381	252	44240	44123·65
272	128	22602	22488·44	327	216	32590	32502·52	382	190	44430	44355·58
273	144	22746	22654·10	328	160	32750	32701·62	383	382	44812	44588·11
274	136	22882	22820·37	329	276	33026	32901·32	384	128	44940	44821·25
275	200	23082	22987·25	330	80	33106	33101·63	385	240	45180	45055·00
276	88	23170	23154·73	331	330	33436	33302·55	386	192	45372	45289·35
277	276	23446	23322·82	332	164	33600	33504·08	387	252	45624	45524·32
278	138	23584	23491·52	333	216	33816	33706·22	388	192	45816	45759·89
279	180	23764	23660·83	334	166	33982	33908·96	389	388	46204	45996·07
280	96	23860	23830·75	335	264	34246	34112·31	390	96	46300	46232·86
281	280	24140	24001·27	336	96	34342	34316·27	391	352	46652	46470·25
282	92	24232	24172·40	337	336	34678	34520·84	392	168	46820	46708·25
283	282	24514	24344·14	338	156	34834	34726·01	393	260	47080	46946·87
284	140	24654	24516·49	339	224	35058	34931·80	394	196	47276	47186·09
285	144	24798	24689·44	340	128	35186	35138·19	395	312	47588	47425·91
286	120	24918	24863·00	341	300	35486	35345·19	396	120	47708	47666·35
287	240	25158	25037·18	342	108	35594	35552·80	397	396	48104	47907·39
288	96	25254	25211·96	343	294	35888	35761·01	398	198	48302	48149·04
289	272	25526	25387·34	344	168	36056	35969·83	399	216	48518	48391·30
290	112	25638	25563·34	345	176	36232	36179·26	400	160	48678	48634·17
291	192	25830	25739·94	346	172	36404	36389·30	401	400	49078	48877·64
292	144	25974	25917·15	347	346	36750	36599·95	402	132	49210	49121·73
293	292	26266	26094·97	348	112	36862	36811·21	403	360	49570	49366·42
294	84	26350	26273·40	349	348	37210	37023·07	404	200	49770	49611·72
295	232	26582	26452·43	350	120	37330	37235·54	405	216	49986	49857·62
296	144	26726	26632·07	351	216	37546	37448·61	406	168	50154	50104·14
297	180	26906	26812·32	352	160	37706	37662·30	407	360	50514	50351·26
298	148	27054	26993·18	353	352	38058	37876·59	408	128	50642	50598·99

of  $(3/\pi^2)n^2$ , but they appear to be pretty accurate. (Note the misprint at  $n=125$ ) — Karl K. Norton.

TABLE (*continued*).

$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$
409	408	51050	50847·33	464	224	65630	65442·14	519	344	82028	81875·93
410	160	51210	51096·27	465	240	65870	65724·52	520	192	82220	82191·75
411	272	51482	51345·83	466	232	66102	66007·51	521	520	82740	82508·18
412	204	51686	51595·99	467	466	66568	66291·11	522	168	82908	82825·21
413	348	52034	51846·76	468	144	66712	66575·31	523	522	83430	83142·85
414	132	52166	52098·14	469	396	67108	66860·13	524	260	83690	83461·10
415	328	52494	52350·12	470	184	67292	67145·55	525	240	83930	83779·95
416	192	52686	52602·72	471	312	67604	67431·58	526	262	84192	84099·42
417	276	52962	52855·92	472	232	67836	67718·22	527	480	84672	84419·49
418	180	53142	53109·73	473	420	68256	68005·46	528	160	84832	84740·17
419	418	53560	53364·15	474	156	68412	68293·32	529	506	85338	85061·46
420	96	53656	53619·17	475	360	68772	68581·78	530	208	85546	85383·36
421	420	54076	53874·80	476	192	68964	68870·85	531	348	85894	85705·87
422	210	54286	54131·04	477	312	69276	69160·52	532	216	86110	86028·98
423	276	54562	54387·89	478	238	69514	69450·81	533	480	86590	86352·70
424	208	54770	54645·35	479	478	69992	69741·70	534	176	86766	86677·03
425	320	55090	54903·42	480	128	70120	70033·20	535	424	87190	87001·97
426	140	55230	55162·09	481	432	70552	70325·31	536	264	87454	87327·51
427	360	55590	55421·39	482	240	70792	70618·03	537	356	87810	87653·66
428	212	55802	55681·26	483	264	71056	70911·35	538	268	88078	87980·42
429	240	56042	55941·76	484	220	71276	71205·29	539	420	88498	88307·79
430	168	56210	56202·86	485	384	71660	71499·83	540	144	88642	88635·77
431	430	56640	56464·57	486	162	71822	71794·98	541	540	89182	88964·35
432	144	56784	56726·89	487	486	72308	72090·73	542	270	89452	89293·54
433	432	57216	56989·82	488	240	72548	72387·10	543	360	89812	89623·34
434	180	57396	57253·36	489	324	72872	72684·07	544	256	90068	89953·75
435	224	57620	57517·50	490	168	73040	72981·65	545	432	90500	90284·77
436	216	57836	57782·26	491	490	73530	73279·84	546	144	90644	90616·39
437	396	58232	58047·62	492	160	73690	73578·63	547	546	91190	90948·62
438	144	58376	58313·58	493	448	74138	73878·04	548	272	91462	91281·46
439	438	58814	58580·16	494	216	74354	74178·05	549	360	91822	91614·91
440	160	58974	58847·34	495	240	74594	74478·67	550	200	92022	91948·97
441	252	59226	59115·14	496	240	74834	74779·90	551	504	92526	92283·64
442	192	59418	59383·54	497	420	75254	75081·73	552	176	92702	92618·91
443	442	59860	59652·54	498	164	75418	75384·18	553	468	93170	92954·79
444	144	60004	59922·16	499	498	75916	75687·23	554	276	93446	93291·28
445	352	60356	60192·38	500	200	76116	75990·89	555	288	93734	93628·38
446	222	60578	60463·22	501	332	76448	76295·15	556	276	94010	93966·08
447	296	60874	60734·66	502	250	76698	76600·03	557	556	94566	94304·39
448	192	61066	61006·70	503	502	77200	76905·52	558	180	94746	94643·31
449	448	61514	61279·36	504	144	77344	77211·61	559	504	95250	94982·84
450	120	61634	61552·62	505	400	77744	77518·31	560	192	95442	95322·98
451	400	62034	61826·49	506	220	77964	77825·62	561	320	95762	95663·72
452	224	62258	62100·97	507	312	78276	78133·54	562	280	96042	96005·07
453	300	62558	62376·06	508	252	78528	78442·06	563	562	96604	96347·03
454	226	62784	62651·75	509	508	79036	78751·19	564	184	96788	96689·60
455	288	63072	62928·05	510	128	79164	79060·93	565	448	97236	97032·77
456	144	63216	63204·97	511	432	79596	79371·28	566	282	97518	97376·55
457	456	63672	63482·48	512	256	79852	79682·23	567	324	97842	97720·94
458	228	63900	63760·61	513	324	80176	79993·79	568	280	98122	98065·94
459	288	64188	64039·35	514	256	80432	80305·96	569	568	98690	98411·55
460	176	64364	64318·69	515	408	80840	80618·74	570	144	98834	98757·76
461	460	64824	64598·64	516	168	81008	80932·13	571	570	99404	99104·58
462	120	64944	64879·20	517	460	81468	81246·12	572	240	99644	99452·01
463	462	65406	65160·36	518	216	81684	81560·72	573	380	100024	99800·05

TABLE (continued).

$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$
574	240	100264	100148·70	629	576	120544	120260·45	684	216	142380	142211·17
575	440	100704	100497·95	630	144	120688	120643·14	685	544	142924	142627·30
576	192	100896	100847·81	631	630	121318	121026·44	686	294	143218	143044·03
577	576	101472	101198·28	632	312	121630	121410·35	687	456	143674	143461·37
578	272	101744	101549·36	633	420	122050	121794·86	688	536	144010	143879·32
579	384	102128	101901·05	634	316	122366	122179·98	689	624	144634	144297·88
580	224	102352	102253·34	635	504	122870	122565·71	690	176	144810	144717·05
581	492	102844	102606·24	636	208	123078	122952·05	691	690	145500	145136·82
582	192	103036	102959·75	637	504	123582	123338·00	692	344	145844	145557·20
583	520	103556	103313·87	638	280	123862	123726·55	693	360	146204	145978·19
584	288	103844	103668·60	639	420	124282	124114·71	694	346	146550	146399·79
585	288	104132	104023·93	640	256	124538	124503·48	695	552	147102	146821·99
586	292	104424	104379·87	641	640	125178	124892·86	696	224	147326	147244·80
587	586	105010	104736·42	642	212	125390	125282·85	697	640	147966	147668·22
588	168	105178	105093·58	643	642	126032	125673·44	698	348	148314	148092·25
589	540	105718	105451·35	644	264	126296	126064·64	699	464	148778	148516·89
590	232	105890	105809·72	645	336	126632	126456·45	700	240	149018	148942·14
591	392	106342	106168·70	646	288	126920	126848·87	701	700	149718	149367·99
592	288	106630	106528·29	647	646	127566	127241·89	702	216	149934	149794·45
593	592	107222	106888·49	648	216	127782	127635·52	703	648	150582	150221·52
594	180	107402	107249·29	649	580	128362	128029·76	704	320	150902	150649·20
595	384	107786	107610·70	650	240	128602	128424·60	705	368	151270	151077·48
596	296	108082	107972·72	651	360	128962	128820·06	706	352	151622	151506·37
597	396	108478	108335·35	652	324	129286	129216·12	707	600	152222	151935·87
598	264	108742	108698·59	653	652	129938	129612·79	708	232	152454	152365·98
599	598	109340	109062·43	654	216	130154	130010·07	709	708	153162	152796·70
600	160	109500	109426·88	655	520	130674	130407·96	710	280	153442	153228·02
601	600	110100	109791·94	656	320	130994	130806·46	711	468	153910	153659·95
602	252	110352	110157·61	657	432	131426	131205·56	712	352	154262	154092·49
603	396	110748	110523·89	658	276	131702	131605·27	713	660	154922	154525·64
604	300	111048	110890·77	659	658	132360	132005·59	714	192	155114	154959·40
605	440	111488	111258·26	660	160	132520	132406·52	715	480	155594	155393·76
606	200	111688	111626·36	661	660	133180	132808·06	716	356	155950	155828·73
607	606	112294	111995·07	662	330	133510	133210·20	717	476	156426	156264·31
608	288	112582	112364·39	663	384	133894	133612·95	718	358	156784	156700·50
609	336	112918	112734·31	664	328	134222	134016·31	719	718	157502	157137·30
610	240	113158	113104·84	665	432	134654	134420·28	720	192	157694	157574·70
611	552	113710	113475·98	666	216	134870	134824·86	721	612	158306	158012·71
612	192	113902	113847·73	667	616	135486	135230·04	722	342	158648	158451·33
613	612	114514	114220·09	668	332	135818	135635·83	723	480	159128	158890·56
614	306	114820	114593·05	669	444	136262	136042·23	724	360	159488	159330·40
615	320	115140	114966·62	670	264	136526	136449·24	725	560	160048	159770·84
616	240	115380	115340·80	671	600	137126	136856·86	726	220	160268	160211·89
617	616	115996	115715·59	672	192	137318	137265·08	727	726	160994	160653·55
618	204	116200	116090·99	673	672	137990	137673·91	728	288	161282	161095·82
619	618	116818	116466·99	674	336	138326	138083·35	729	486	161768	161538·69
620	240	117058	116843·60	675	360	138686	138493·40	730	288	162056	161982·17
621	396	117454	117220·82	676	312	138998	138904·05	731	672	162728	162426·26
622	310	117764	117598·65	677	676	139674	139315·31	732	240	162968	162870·96
623	528	118292	117977·08	678	224	139898	139727·18	733	732	163700	163316·27
624	192	118484	118356·12	679	576	140474	140139·66	734	366	164066	163762·18
625	500	118984	118735·77	680	256	140730	140552·75	735	336	164402	164208·70
626	312	119296	119116·03	681	452	141182	140966·44	736	352	164754	164655·83
627	360	119656	119496·90	682	300	141482	141380·74	737	660	165414	165103·57
628	312	119968	119878·37	683	682	142164	141795·65	738	240	165654	165551·92

For  $n=688$ ,  $\tau(n)=336$ .

TABLE (*continued*).

$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$
739	738	166392	166000·87	794	396	191870	191629·56	849	564	219340	219097·23
740	288	166680	166450·43	795	416	192286	192112·56	850	320	219660	219613·66
741	432	167112	166900·60	796	396	192682	192596·17	851	792	220452	220130·71
742	312	167424	167351·38	797	796	193478	193080·39	852	280	220732	220648·36
743	742	168166	167802·77	798	216	193694	193565·21	853	852	221584	221166·62
744	240	168406	168254·76	799	736	194430	194050·64	854	360	221944	221685·48
745	592	168998	168707·36	800	320	194750	194536·67	855	432	222376	222204·96
746	372	169370	169160·57	801	528	195278	195023·32	856	424	222800	222725·04
747	492	169862	169614·39	802	400	195678	195510·57	857	856	223656	223245·73
748	320	170182	170068·82	803	720	196398	195998·43	858	240	223896	223767·03
749	636	170818	170523·85	804	264	196662	196486·90	859	858	224754	224288·93
750	200	171018	170979·50	805	528	197190	196975·98	860	336	225090	224811·44
751	750	171768	171435·75	806	360	197550	197465·66	861	480	225570	225334·56
752	368	172136	171892·61	807	536	198086	197955·96	862	430	226000	225858·29
753	500	172636	172350·07	808	400	198486	198446·86	863	862	226862	226382·62
754	336	172972	172808·14	809	808	199294	198938·37	864	288	227150	226907·57
755	600	173572	173266·82	810	216	199510	199430·48	865	688	227838	227433·12
756	216	173788	173726·11	811	810	200320	199923·21	866	432	228270	227959·28
757	756	174544	174186·01	812	336	200656	200416·54	867	544	228814	228486·05
758	378	174922	174646·52	813	540	201196	200910·48	868	360	229174	229012·43
759	440	175362	175107·63	814	360	201556	201405·03	869	780	229954	229541·41
760	288	175650	175569·35	815	648	202204	201900·19	870	224	230178	230070·01
761	760	176410	176031·68	816	256	202460	202395·95	871	792	230970	230599·21
762	252	176662	176494·62	817	756	203216	202892·32	872	432	231402	231129·02
763	648	177310	176958·16	818	408	203624	203389·30	873	576	231978	231659·43
764	380	177690	177422·31	819	432	204056	203886·89	874	396	232374	232190·46
765	384	178074	177887·07	820	320	204376	204385·09	875	600	232974	232722·09
766	382	178456	178352·44	821	820	205196	204883·89	876	288	233262	233254·33
767	696	179152	178818·42	822	272	205468	205383·30	877	876	234138	232787·18
768	256	179408	179285·00	823	822	206290	205883·32	878	438	234576	234320·64
769	768	180176	179752·19	824	408	206698	206383·95	879	584	235160	234854·70
770	240	180416	180219·99	825	400	207098	206885·19	880	320	235480	235389·37
771	512	180928	180688·40	826	348	207446	207387·03	881	880	236360	235924·65
772	384	181312	181157·42	827	826	208272	207889·48	882	252	236612	236460·54
773	772	182084	181627·04	828	264	208536	208392·54	883	882	237494	236997·04
774	252	182336	182097·27	829	828	209364	208896·21	884	384	237878	237534·14
775	600	182936	182568·11	830	328	209692	206400·49	885	464	238342	238071·85
776	384	183320	183039·56	831	552	210244	209905·37	886	442	238784	238610·17
777	432	183752	183511·61	832	384	210628	210410·86	887	886	239670	239149·10
778	388	184140	183984·28	833	672	211300	210916·96	888	288	239958	239688·64
779	720	184860	184457·55	834	276	211576	211423·67	889	756	240714	240228·78
780	192	185052	184931·43	835	664	212240	211930·98	890	352	241066	240769·53
781	700	185752	185405·92	836	360	212600	212438·91	891	540	241606	241310·89
782	352	186104	185881·01	837	540	213140	212947·44	892	444	242050	241852·86
783	504	186608	186356·71	838	418	213558	213456·58	893	828	242878	242395·43
784	336	186944	186833·02	839	838	214396	213966·32	894	296	243174	242938·62
785	624	187568	187309·94	840	192	214588	214476·68	895	712	243886	243482·41
786	260	187828	187787·47	841	812	215400	214987·64	896	384	244270	244026·81
787	786	188614	188265·60	842	420	215820	215499·21	897	528	244798	244571·81
788	392	189006	188744·34	843	560	216380	216011·39	898	448	245246	245117·43
789	524	189530	189223·69	844	420	216800	216524·18	899	840	246086	245663·65
790	312	189842	189703·65	845	624	217424	217037·57	900	240	246326	246210·48
791	672	190514	190184·22	846	276	217700	217551·58	901	832	247158	246757·91
792	240	190754	190665·39	847	660	218360	218066·19	902	400	247558	247305·96
793	720	191474	191147·17	848	416	218776	218581·40	903	504	248062	247854·61

TABLE\* (*continued*).

$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$	$n$	$\tau(n)$	$T(n)$	$\frac{3}{\pi^2} n^2$
904	448	248510	248403·88	937	936	267256	266870·57	970	384	286076	285999·30
905	720	249230	248953·75	938	396	267652	267440·51	971	970	287046	286589·30
906	300	249530	249504·22	939	624	268276	268011·05	972	324	287370	287179·90
907	906	250436	250055·31	940	368	268644	268582·19	973	828	288198	287771·11
908	452	250888	250607·00	941	940	269584	269153·95	974	486	288684	288362·92
909	600	251488	251159·31	942	312	269896	269726·31	975	480	289164	288955·35
910	288	251776	251712·22	943	880	270776	270299·28	976	480	289644	289548·39
911	910	252686	252265·73	944	464	271240	270872·86	977	976	290620	290142·03
912	288	252974	252819·86	945	432	271672	271447·05	978	324	290944	290736·28
913	820	253794	253374·59	946	420	272092	272021·84	979	880	291824	291331·13
914	456	254250	253929·93	947	946	273038	272597·25	980	336	292160	291926·60
915	480	254730	254485·88	948	312	273350	273173·26	981	648	292808	292522·67
916	456	255186	255042·44	949	864	274214	273749·88	982	490	293298	293119·35
917	780	255966	255599·61	950	360	274574	274327·10	983	982	294280	293716·64
918	288	256254	256157·38	951	632	275206	274905·94	984	320	294600	294314·54
919	918	257172	256715·76	952	384	275590	275483·38	985	784	295384	294913·04
920	352	257524	257274·75	953	952	276542	276062·43	986	448	295832	295512·15
921	612	258136	257834·34	954	312	276854	276642·09	987	552	296384	296111·87
922	460	258596	258394·55	955	760	277614	277222·36	988	432	296816	296712·20
923	840	259436	258955·36	956	476	278090	277803·23	989	924	297740	297313·14
924	240	259676	259516·78	957	560	278650	278384·71	990	240	297980	297914·68
925	720	260396	260078·81	958	478	279128	278966·80	991	990	298970	298516·83
926	462	260858	260641·45	959	816	279944	279549·50	992	480	299450	299119·59
927	612	261470	261204·69	960	256	280200	280132·81	993	660	300110	299722·96
928	448	261918	261768·55	961	930	281130	280716·72	994	420	300530	300326·94
929	928	262846	262333·01	962	432	281562	281301·24	995	792	301322	300931·52
930	240	263086	262898·07	963	636	282198	281886·37	996	328	301650	301536·71
931	756	263842	263463·75	964	480	282678	282472·11	997	996	302646	302142·51
932	464	264306	264030·03	965	768	283446	283058·46	998	498	303144	302748·92
933	620	264926	264596·93	966	264	283710	283645·41	999	648	303792	303355·93
934	466	265392	265164·43	967	966	284676	284232·97	1000	400	304192	303963·55
935	640	266032	265732·53	968	440	285116	284821·14				
936	288	266320	266301·25	969	576	285692	285409·92				

\* In the extended as well as in the original Table it will be seen that the sum-totient is always intermediate between  $3/\pi^2 \cdot n^2$  and  $3/\pi^2 \cdot (n+1)^2$ .

The formula of verification applied at every tenth step to the  $T$  column precludes the possibility of the existence of other than typographical errors or errors of transcription. Accumulative errors are rendered impossible.

## 10.

### ON THE EQUATION TO THE SECULAR INEQUALITIES IN THE PLANETARY THEORY.

[*Philosophical Magazine*, xvi. (1883), pp. 267—269.]

A very long time ago I gave, in this *Magazine*\*, a proof of the reality of the roots in the above equation, in which I employed a certain property of the square of a symmetrical matrix which was left without demonstration. I will now state a more general theorem concerning the *product* of *any* two matrices of which that theorem is a particular case. In what follows it is of course to be understood that the product of two matrices means the matrix corresponding to the *combination* of two *substitutions* which those matrices represent.

It will be convenient to introduce here a notion (which plays a conspicuous part in my new theory of multiple algebra), namely that of the *latent roots* of a matrix—latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf. If from each term in the diagonal of a given matrix,  $\lambda$  be subtracted, the determinant to the matrix so modified will be a rational integer function of  $\lambda$ ; the roots of that function are the latent roots of the matrix; and there results the important theorem that the latent roots of any function of a matrix are respectively the same functions of the latent roots of the matrix itself: for example, the latent roots of the square of a matrix are the squares of its latent roots.

The latent roots of the product of two matrices, it may be added, are the same in whichever order the factors be taken. If, now,  $m$  and  $n$  be any two matrices, and  $M = mn$  or  $nm$ , I am able to show that the sum of the products of the latent roots of  $M$  taken  $i$  together in every possible way is equal to the sum of the products obtained by multiplying every minor determinant of the  $i$ th order in one of the two matrices  $m, n$  by its *altruistic opposite* in the other: the reflected image of any such determinant, in respect to the principal diagonal of the matrix to which it belongs, is its *proper opposite*, and the corresponding determinant to this in the other matrix is its *altruistic opposite*.

[\* Vol. i. of this Reprint, p. 378.]

The proof of this theorem will be given in my large forthcoming memoir on Multiple Algebra designed for the *American Journal of Mathematics*.

Suppose, now, that  $m$  and  $n$  are transverse to one another, that is, that the lines in the one are identical with the columns in the other, and *vice versa*, then any determinant in  $m$  becomes identical with its altruistic opposite in  $n$ ; and furthermore, if  $m$  be a symmetrical matrix, it is its own transverse. Consequently we have the theorem (the one referred to at the outset of this paper) that the sum of the  $i$ -ary products of the latent roots of the square of a symmetrical matrix (that is, of the squares of the roots of the matrix itself) is equal to the sum of the squares of all the minor determinants of the order  $i$  in the matrix; whence it follows, from Descartes's theorem, that when all the terms of a symmetrical matrix are real, none of its latent roots can be *pure imaginaries*, and, as an easy inference, cannot be *any kind* of imaginaries; or, in other words, all the latent roots of a symmetrical matrix are real, which is Laplace's theorem.

I may take this opportunity of stating the important theorem that if  $\lambda_1, \lambda_2, \dots, \lambda_i$  are the latent roots of any matrix  $m$ , then

$$\phi m = \sum \frac{(m - \lambda_2)(m - \lambda_3) \dots (m - \lambda_i)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \dots (\lambda_1 - \lambda_i)} \phi \lambda.$$

This theorem of course presupposes the rule first stated by Prof. Cayley (*Phil. Trans.* 1857) for the addition of matrices.

When any of the latent roots are equal, the formula must be replaced by another obtained from it by the usual method of infinitesimal variation. If

$\overset{1}{\phi} m = m^{\omega}$ , it gives the expression for the  $\omega$ th root of the matrix; and we see that the number of such roots is  $\omega^i$ , where  $i$  is the order of the matrix. When, however, the matrix is *unitary*, that is, all its terms except the diagonal ones are *zeros*, or *zeroidal*, that is, when all its terms are *zeros*, this conclusion is no longer applicable, and a certain definite number of arbitrary quantities enter into the general expressions for the roots.

The case of the extraction of any root of a unitary matrix of the second order was first considered and successfully treated by the late Mr Babbage; it reappears in M. Serret's *Cours d'Algèbre supérieure*. This problem is of course the same as that of finding a function  $\frac{ax+b}{cx+d}$  of any given order of periodicity. My memoir will give the solution of the corresponding problem for a matrix of any order. Of the many unexpected results which I have obtained by my new method, not the least striking is the *rapprochement* which it establishes between the theory of Matrices and that of Invariants. The theory of invariance relative to associated Matrices includes and transcends that relative to algebraical functions.

# 11.

## ON THE INVOLUTION AND EVOLUTION OF QUATERNIONS.

[*Philosophical Magazine*, xvi. (1883), pp. 394—396.]

THE subject-matter of quaternions is really nothing more nor less than that of substitutions of the second order, such as occur in the familiar theory of quadratic forms. A linear substitution of the second order is in essence identical with a square matrix of the second order, the law of multiplication between one such matrix and another being understood to be the same as that of the composition of one substitution with another, and therefore depending on the order of the factors; but as regards the multiplication of three or more matrices, subject to the same associative law as in ordinary algebraical multiplication.

Every matrix of the second order may be regarded as representing a quaternion, and *vice versa*; in fact if, using  $i$  to denote  $\sqrt{(-1)}$ , we write a matrix  $m$  of the second order under the form

$$\begin{array}{ll} a+bi, & c+di, \\ -c+di, & a-bi, \end{array}$$

we have by definition,

$$m = a\alpha + b\beta + c\gamma + d\delta,$$

$$\text{where } \alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

$$\begin{aligned} \text{Now } \alpha^2 &= \alpha, & \beta^2 &= \gamma^2 = \delta^2 = -\alpha, \\ \alpha\beta &= \beta\alpha = \beta, & \alpha\gamma &= \gamma\alpha = \gamma, & \alpha\delta &= \delta\alpha = \delta, \\ \beta\gamma &= -\gamma\beta = \alpha, & \gamma\delta &= -\delta\gamma = \beta, & \delta\beta &= -\beta\delta = \gamma; \end{aligned}$$

so that we may for  $\alpha, \beta, \gamma, \delta$ , substitute  $1, h, k, l$ , four symbols subject to the same laws of self-operation and mutual interaction as unity and the three Hamiltonian symbols. Now I have given the universal formula for expressing any given function of a matrix of *any* order as a rational function of that matrix and its latent roots; and consequently the  $q$ th power or root of any

quadratic matrix, and therefore of any quaternion, is known. As far as I am informed, only the square root of a quaternion has been given in the textbooks on quaternions, notably by Hamilton in his Lectures on Quaternions.

The latent roots of  $m$  are the roots of the quadratic equation

$$\lambda^2 - 2a\lambda + a^2 + b^2 + c^2 + d^2 = 0.$$

The general formula

$$\phi m = \Sigma \phi \lambda_1 \frac{(m - \lambda_2)(m - \lambda_3) \dots (m - \lambda_i)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \dots (\lambda_1 - \lambda_i)},$$

where  $i$  is the order of the matrix  $m$ , when  $i = 2$  and  $\phi m = m^{\frac{1}{q}}$ , becomes

$$m^{\frac{1}{q}} = \frac{\lambda_1^{\frac{1}{q}} - \lambda_2^{\frac{1}{q}}}{\lambda_1 - \lambda_2} m - \frac{\lambda_2 \lambda_1^{\frac{1}{q}} - \lambda_1 \lambda_2^{\frac{1}{q}}}{\lambda_1 - \lambda_2},$$

where  $\lambda_1, \lambda_2$  are the roots of the above equation. If  $\mu$  is the modulus of the quaternion, namely is  $\sqrt{a^2 + b^2 + c^2 + d^2}$ , and  $\mu \cos \theta = a$ , the latent roots  $\lambda_1, \lambda_2$  assume the form

$$\mu (\cos \theta \pm i \sin \theta).$$

When the modulus is zero the two latent roots are equal to one another, and to  $a$ , the scalar of the quaternion; so that in this case the ordinary theory of vanishing fractions shows that

$$m^{\frac{1}{q}} = a^{\frac{1}{q}} \left( \frac{m}{a} + \frac{q-1}{q} \right).$$

In the general case there are  $q^2$  roots of the  $q$ th order to a quaternion. Calling

$\frac{\pi}{q} = \omega$ , and writing  $m^{\frac{1}{q}} = Am + B$ ,

$$A = \frac{\mu^{\frac{1}{q}} \cos \left( \frac{\theta}{q} + 2k\omega \right) + i \sin \left( \frac{\theta}{q} + 2k\omega \right) - \cos \left( \frac{\theta}{q} + 2k'\omega \right) + i \sin \left( \frac{\theta}{q} + 2k'\omega \right)}{2i \sin \theta},$$

$$B = -\mu^{\frac{1}{q}} \frac{\cos \left( \frac{q-1}{q} \theta + 2k\omega \right) + i \sin \left( \frac{q-1}{q} \theta + 2k\omega \right) - \cos \left( \frac{q-1}{q} \theta + 2k'\omega \right) + i \sin \left( \frac{q-1}{q} \theta + 2k'\omega \right)}{2i \sin \theta}.$$

For the  $q$  system of values  $k = k' = 1, 2, 3 \dots q$ , the coefficients  $A$  and  $B$  will be real, for the other  $q^2 - q$  systems of values imaginary; so that there are  $q$  quaternion-proper  $q$ th roots of a quaternion-proper in Hamilton's sense, and  $q^2 - q$  of the sort which, by a most regrettable piece of nomenclature, he terms bi-quaternions. The real or proper-quaternion values of  $m^{\frac{1}{q}}$  are

$$\frac{\mu^{\frac{1}{q}}}{\sin \theta} \left\{ \sin \left( \frac{\theta}{q} + 2k\omega \right) \frac{m}{\mu} + \sin \left( \frac{q-1}{q} \theta + 2k\omega \right) \right\},$$

$\mu^{\frac{1}{q}}$  meaning *the* or (when there is an alternative) *either* real value of the  $q$ th root of the modulus.

In the  $q$ th root (or power) of a quaternion  $m$ , the form  $Am + B$  shows that the vector-part remains constant to an ordinary algebraical factor *près*; and we know *a priori* from the geometrical point of view that this ought to be the case. When the vector disappears a porism starts into being; and besides the values of the roots given by the general formula, there are others involving arbitrary parameters. Babbage's famous investigation of the form of the homographic function of  $\frac{px+q}{rx+s}$  of  $x$ , which has a periodicity of any given degree  $q$ , is in fact (surprising as such a statement would have appeared to Babbage and Hamilton) one and the same thing as to find the  $q$ th root of unity under the form of a quaternion!

It is but justice to the eminent President of the British Association to draw attention to the fact that the substance of the results here set forth (although arrived at from an independent and more elevated order of ideas) may be regarded as a statement (reduced to the explicit and most simple form) of results capable of being extracted from his memoir on the Theory of Matrices, *Phil. Trans.* Vol. CXLVIII. (1858) (*vide* pp. 32—34, arts. 44—49).

## 12.

### ON THE INVOLUTION OF TWO MATRICES OF THE SECOND ORDER.

[*British Association Report, Southport (1883)*, pp. 430—432.]

If  $m, n$  be two matrices of any order  $i$ , then, taking the determinant of the matrix  $z + yn + xm$ , there results a ternary quantic in the variables  $x, y, z$ , which may be termed the quantic of the corpus  $m, n$ .

In what follows I confine myself almost exclusively to the case of a corpus of the second order; the quantic may be written

$$z^2 + 2bzx + 2cyz + dx^2 + 2exy + fy^2:$$

it is then easy to establish the identical relations

$$\begin{aligned}m^2 - 2bm + d &= 0, \\mn + nm - 2bn - 2cm + 2e &= 0, \\n^2 - 2cn + f &= 0.\end{aligned}$$

It hence easily appears that any given function of  $m, n$  can, by aid of the five parameters  $b, c, d, e, f$ , be expressed in the form  $A + Bm + Cn + Dmn$ .

This form containing four arbitrary constants, it follows that in general any given matrix of the second order can be expressed as a function of  $m$  and  $n$ ; for there will be four linear equations between  $A, B, C, D$  and the four elements of the given matrix. But this statement is subject to two cases of exception.

The first of these is when  $n$  and  $m$  are functions of one another: for in this case  $A + Bm + Cn + Dmn$  is reducible to the form  $P + Qm$ , and there will be only two disposable constants wherewith to satisfy the four linear equations.

The second case is when the determinant of the fourth order formed by the elements of the four matrices  $\begin{array}{|cc|} 1, & m \\ n, & mn \end{array}$  vanishes; writing

$$m, n = \begin{vmatrix} t_1, & t_2 \\ t_3, & t_4 \end{vmatrix}, \quad \begin{vmatrix} \tau_1, & \tau_2 \\ \tau_3, & \tau_4 \end{vmatrix}$$

respectively, it is not difficult to show that the value of this determinant is

$$(t_2\tau_3 - \tau_2 t_3)^2 + \{(t_1 - t_4)\tau_2 - (\tau_1 - \tau_4)t_2\} \{(t_1 - t_4)\tau_3 - (\tau_1 - \tau_4)t_3\}.$$

This expression is a function of the five parameters  $b, c, d, e, f$ , as may be shown in a variety of ways.

Thus it is susceptible of easy proof that if  $\mu_1, \mu_2$  are the roots of the equation  $\mu^2 - 2b\mu + d = 0$ , and  $\nu_1, \nu_2$  the roots of the equation  $\nu^2 - 2c\nu + f = 0$ , then, the two matrices being related as above, we must have

$$\begin{aligned}(m - \mu_1)(n - \nu_1) &= 0, \\ (n - \nu_2)(m - \mu_2) &= 0,\end{aligned}$$

and consequently, by virtue of the middle one of the three identities,

$$\mu_1\nu_1 + \mu_2\nu_2 - 2e = 0.$$

Writing this in the form

$$(\mu_1\nu_1 + \mu_2\nu_2 - 2e)(\mu_1\nu_2 + \mu_2\nu_1 - 2e) = 0,$$

$$\text{this is } 4e^2 - 2e \cdot 4bc + (\mu_1^2 + \mu_2^2)\nu_1\nu_2 + (\nu_1^2 + \nu_2^2)\mu_1\mu_2 = 0,$$

$$\text{which gives } e^2 - 2bce + b^2f + c^2d - df = 0;$$

the function on the left hand is the invariant (discriminant) of the ternary quantic appurtenant to the corpus, and we have this invariant = 0 as the necessary and sufficient condition of the involution of the elements of the corpus ; the invariant in question is for this reason called the involutant.

Expressing the values of the coefficients in terms of the elements of the two matrices, namely

$$\begin{aligned}2b &= t_1 + t_4, & 2c &= \tau_1 + \tau_4, \\ d &= t_1t_4 - t_2t_3, & 2e &= t_1\tau_4 + \tau_1t_4 - t_2\tau_3 - t_3\tau_2, & f &= \tau_1\tau_4 - \tau_2\tau_3,\end{aligned}$$

it at once appears that the two expressions for the involutant are, to a numerical factor *près*, identical.

It can be shown *à priori* that the involutant of a corpus of the second order must be expressible in terms of the coefficients of the function ; and therefore, being obviously invariantive in regard to linear substitutions impressed on  $m, n$ , it must be also invariantive for linear substitutions impressed on  $z, x, y$ , and must therefore be the invariant of the function. The corresponding theorem is not true, it should be observed, for the involutant of a corpus beyond the second order ; for such involutant cannot in general be expressed in terms of the coefficients of the function.

The expression for the involutant in terms of the  $t$ 's and  $\tau$ 's may also be obtained directly from the equation  $(m - \mu_1)(n - \nu_1) = 0$ . To this end it is only necessary to single out any term of the matrix represented by the left-hand side of the equation and equate it to zero : the resulting equation rationalised will be found to reproduce the expression in question.

I have thus indicated four methods of obtaining the involutant to a matrix-corpus of the second order ; but there is yet a fifth, the simplest of all, and the most suggestive of the course to be pursued in investigating the higher order of involutants.

I observe that for a corpus of any order the function  $mn - nm$  is invariantive for any linear substitution impressed on  $m$  and  $n$ . Its determinant will therefore be an invariant for any substitution impressed on  $m$  and  $n$ . When  $m$  and  $n$  are of the second order, reducing each term of  $(mn - nm)^2$ , that is  $mnmn - mn^2m - nm^2n + nmnm$ , and of  $mn - nm$ , by means of the three identical equations, to the form of a linear function of  $mn, m, n, 1$ , it will be found without difficulty that there results the identical equation

$$(mn - nm)^2 + I = 0,$$

the coefficient of  $mn - nm$  vanishing. Consequently the determinant of the matrix  $mn - nm$  is equal to  $I$ , which on calculation will be found to be identical with the invariant of the ternary quadric function.

It is obvious from the three identical equations that if  $m, n$  are in involution—that is, if their involutant is zero—every rational and integral function of  $m, n$  will be in involution with every other rational and integral function of  $m, n$ . Hence follows this new and striking theorem concerning matrices of the second order: If  $f(m, n)$  and  $\phi(m, n)$  are any rational functions whatever of  $m, n$ , the determinant to the matrix  $mn - nm$  is contained as a factor in the determinant to the matrix  $f\phi - \phi f$ .

It may be noticed that  $f, \phi$  need not be integer functions *by stipulation*, because any linear function of  $mn, m, n, 1$ , divided anteriorly or posteriorly by a second like function, can itself be expressed as a linear function of the same four terms.

As a very simple example of the theorem, observe that the determinant of  $m^2n - nm^2$  will contain as a factor the determinant of  $mn - nm$ .