VII.

ON THE PRINCIPAL AXES OF A RIGID BODY*.

THE object of this note is to simplify the manner in which the theory of principal axes is made to depend on the theory of confocal surfaces of the second order.

It is well known†, that if A, B, C are the principal moments of inertia of a rigid body at the centre of gravity, the moment of inertia about an axis through the centre of gravity, whose direction-cosines referred to the principal axes are l, m, n is

$$l^2A + m^2B + n^2C \qquad (1).$$

But if M be the mass of the body, and we draw the ellipsoid

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \frac{1}{M} \qquad (2),$$

(ellipsoid of gyration), this amounts to saying that the moment of inertia is Mp^2, where p is the perpendicular from the centre on the tangent plane

$$lx + my + nz = p \qquad (3),$$

of the ellipsoid (2); that is, if we draw a plane perpendicular to the given axis to touch the ellipsoid (2), then p is the central perpendicular on this plane.

It is also known‡ that the moment of inertia about any

* [From *The Oxford, Cambridge and Dublin Messenger of Mathematics*, Vol. IV. pp. 78—81.]

† Routh's *Rigid Dynamics*, p. 7.

‡ Routh, *l. c.* p. 2.

axis whatever, is equal to the moment about a parallel axis through the centre of gravity, together with the moment which the whole mass, if collected at the centre of gravity, would have about the original axis.

These things being so, the following is a construction for the moment of inertia about any axis PQ through a point P. Draw a plane PT through P perpendicular to the axis; then determine λ so that the surface

$$\frac{x^2}{A+\lambda M}+\frac{y^2}{B+\lambda M}+\frac{z^2}{C+\lambda M}=\frac{1}{M},$$

may touch this plane PT; this gives a simple equation* for λ, which has consequently only one value. Then the required moment of inertia is $M(OP^2-\lambda)$, O being the centre of gravity, which is the origin.

For, draw OT perpendicular to the plane PT; and let Ot be the perpendicular on the parallel plane which touches the ellipsoid of gyration. Then OT being a parallel axis through the centre of gravity, and PT the perpendicular distance of O from PQ; the moment of inertia about PQ is

$$M(Ot^2+PT^2).$$

But† $$OT^2=l^2\left(\frac{A}{M}+\lambda\right)+m^2\left(\frac{B}{M}+\lambda\right)+n^2\left(\frac{C}{M}+\lambda\right)$$
$$=\frac{l^2A+m^2B+n^2C}{M}+\lambda$$
$$=Ot^2+\lambda.$$

Hence $$Ot^2+PT^2=OT^2+PT^2-(OT^2-Ot^2)$$
$$=OP^2-\lambda,$$

the moment of inertia is therefore $M(OP^2-\lambda)$.

Now, consider the confocal ellipsoid which passes through P, and its tangent plane (A) at P. Let λ_1 be the value of λ for this ellipsoid. Then if the plane A be turned ever so little round P, it will begin to cut the ellipsoid in a small ellipse;

* Salmon's *Geometry of three dimensions*, p. 121.

† Salmon, *l. c.* p. 58.

and so the confocal ellipsoid which touches the plane A in its new position, will lie wholly within the other. Its axes will therefore be less than those of the other, and therefore λ will be less than λ_1. That is to say, the value of λ is a *maximum* for the plane which touches the confocal ellipsoid through P. Therefore the moment of inertia about the axis perpendicular to this plane, namely $M(OP^2 - \lambda_1)$, is a *minimum*. Now the axis of least moment at any point is a principal axis. It follows therefore, that *the normal to the confocal ellipsoid through* P *is the principal axis of least moment at* P.

In a manner precisely similar, it may be shewn that if we draw through P an hyperboloid of two sheets confocal to the ellipsoid of gyration, the value of λ for its tangent plane at P is a *minimum;* and therefore that the normal to this surface at P is the principal axis of greatest moment at P.

This being so, we know that the remaining principal axis is perpendicular to these two, and is therefore normal to the confocal hyperboloid of one sheet which passes through P.

We have proved then that the principal axes at any point P, are the normals to the three surfaces confocal to the ellipsoid of gyration which pass through P; and if λ, μ, ν are the values of λ for these three surfaces, or, as we may say, if λ, μ, ν are the curvilinear co-ordinates of P in respect of the ellipsoid of gyration, then the three moments of inertia are

$$M(OP^2 - \lambda), \quad M(OP^2 - \mu), \quad M(OP^2 - \nu).$$

This connection having been established, all the usual propositions about principal axes follow at once from the known theory of confocal surfaces. Thus, the locus of points where two of the principal moments are equal, is the focal conics of the ellipsoid of gyration. For*, of the three confocal surfaces which pass through any point on a focal conic, two coincide with the focal conic itself. For any point on the focal ellipse, the momental ellipsoid is an oblate spheroid; and for

* Salmon, *l. c.* p. 114.

any point on the focal hyperbola, the momental ellipsoid is a prolate spheroid.

Again*, two confocal surfaces can be drawn to touch any given straight line, and the two corresponding tangent planes are at right angles. Hence, on every straight line there are two points at which one principal axis is perpendicular to the line, and these two axes are at right angles. If the two points of contact coincide, then at that point two principal axes are perpendicular to the line, which is therefore itself a principal axis. The condition, therefore, that an axis may be a principal axis at some point of its length, is that the two points of contact of confocal surfaces touching it must coincide; which is obvious, for in that case the line is normal to the third surface passing through the common point of contact.

* Salmon, *l. c.* p. 127.

VIII.

SYNTHETIC PROOF OF MIQUEL'S THEOREM*.

In the note to page 235 of Dr Salmon's *Conics*, mention is made of a theorem originally given by M. Auguste Miquel, in Liouville's *Journal*, Vol. x., p. 349. The theorem may be stated as follows. It is known that we can draw exactly one parabola to touch four given lines. If now we have five lines given, we can draw five parabolas, each of which touches four of the given lines. The theorem is that the foci of these five parabolas lie all on one circle. M. Miquel's proof, reproduced by Catalan, depends on the fact that the circle circumscribing the triangle formed by three tangents to a parabola passes through the focus. Since, as above remarked, a parabola can be drawn to touch any four straight lines, it follows from this that the four circles, circumscribing the four triangles which we get by leaving out each of the lines in turn, all meet in a point, the focus of the parabola. The theorem can thus be stated as a property of straight lines and circles, without any mention of parabolas; and accordingly M. Miquel proves it by using the ordinary (Euclid) geometry appropriate to such theorems.

I find, however, that the theorem connects itself in a somewhat interesting way with that general conception of geometrical facts which regards all the properties of a curve or other continuous figure as depending upon its order alone, and which it has become usual to call Synthetic Geometry. I am not

* [From *The Oxford, Cambridge and Dublin Messenger of Mathematics*, Vol. v. pp. 124—141.]

satisfied with the word, and use it under protest for want of a better. The *thing* should be more widely known than it is in this country; the simplicity and instructiveness of its application in the present case are my reasons for calling attention to a matter otherwise unimportant. Those who are of the mind of Herr Gretschel may substitute for the words "Synthetic Proof," "Proof by *Organic* Geometry."

I.

On the Shapes of Certain Curves.

It is very important that we should know what we mean by a *curve*. To this end I start with Plücker's mode of generating curves.

Imagine a point and a straight line passing through it to be moved about on a plane in this manner. The point always moves along the line, never stopping for any portion of time however small; and all this while the line is turning round the point and never stops for any portion of time however small. Then the point will *describe*, and the line will *envelope*, a curve. To get a clear notion of this, take the particular case of a line being rolled round a circle (without sliding) so as always to touch it. The point of contact will then be always moving along the tangent, and the tangent will always be turning round the point of contact. But, at the same time, the point will by its motion have traced out the circle; and if we imagine all that part of the plane which the line passes over to become black, there will be left a white patch bounded by this circle, *enveloped* by all the positions of the moving line.

Now here there are three things to consider:—

A. The actual curve which you see, dividing that portion of the plane which is inside it from the portion which is outside it.

B. The assemblage of all the positions of the moving point.

C. The assemblage of all the positions of the moving line.

It is very easy but at the same time very important to observe that these are three distinct things. We are accustomed to say that *A* is the *locus* of *B* (the points) and the *envelope* of *C* (the tangents). Every curve of course has an assemblage of points upon it, and an assemblage of lines touching it; but it is not *the same thing* as either of these, any more than the assemblage of points is the same thing as the assemblage of lines. I shall illustrate this further by taking the two simplest and most fundamental cases.

Namely, suppose first that the point, instead of moving, remains always at rest while the line turns round it. Then the figure *A* is merely the point itself and no longer a curve. *B* has entirely disappeared; there is no assemblage of positions of the moving point. For by an *assemblage* of positions [of a point] we mean at least a line; now a point is absolutely *no* line, as a line is *no* surface, and a surface *no* space. *C* is now the assemblage of lines through the point. So then of our three things *A* and *C* remain, while *B* has entirely disappeared; but it is exceedingly obvious that a point and the assemblage of straight lines through it are two different things. Next suppose that the line remains still while the point moves along it. Then *A* is the straight line, which we are not accustomed to call a curve. *B* is the assemblage of all the points on this line. *C* has entirely disappeared, for there is no assemblage of lines. But a straight line is not the same thing as all the points on it, though you may think so at first. To be convinced, contemplate the other case just considered; you have just as much right to say that a point is the same thing as all the lines through it. And then read S. Thomas Aquinas on this question, if you can find the reference, which I have forgotten.

An assemblage of points is said to be of a certain *order*, when a certain number of the points can be found upon an arbitrary straight line. Thus the assemblage of points lying upon a straight line is of the *first* order, because *one* of the points can be found upon another arbitrary straight line.

The assemblage of points lying upon two straight lines is of the second order, since two of them can be found upon another arbitrary straight line. And generally the assemblage of all the points lying upon n straight lines is of the n^{th} order.

Similarly, an assemblage of lines is said to be of a certain *class*, when a certain number of the lines can be drawn through an arbitrary point. Thus the assemblage of lines passing through a point is of the *first* class, because *one* of the lines can be drawn through another arbitrary point. The assemblage of lines passing through two points is of the second class, because two of the lines can be drawn through another arbitrary point. And generally the assemblage of all the lines passing through n points is of the n^{th} class.

We have now a number appertaining to the aggregate of points, viz. its *order*, and a number appertaining to the aggregate of lines, viz. its *class*. Neither of these numbers belong in strictness to the curve itself; but there is a number—the *Geschlecht-zahl* or *deficiency*—which does belong to the curve, and not immediately to the points or tangents*. It is not however my business to speak of that now. I am going to investigate certain cases in which these two things—the assemblage of points, and the assemblage of tangents—change continuously together; and in which it is very important to observe the modifications which *both* of them undergo.

A curve of the third class is, in general, of the shape represented in fig. 1; that is to say, it consists of a tricusp surrounded by an oval. It is very easy to see that from any point within the tricusp we can draw three tangents to it and none to the oval; from any point in the intermediate space we can draw one tangent to the tricusp and none to the oval; and from any point outside the oval we can draw two tangents to it and one to the tricusp. Such a curve is of the sixth order; and it is singled out among curves of the

* [This incidental remark seems noteworthy: the number in question belongs as much, and in the same way, to the tangents as to the points, that is, not peculiarly to either: but the author's point of view seems to be a different one. C.]

sixth order as having nine cusps, of which as you see only three are real. The tangents at three real cusps always meet in a point*.

Here then is an assemblage of points which is of the sixth order connected with an assemblage of lines which is of the third class by the fact that they are respectively the points and tangents of a certain curve. I am going to alter the curve, and to watch what becomes of these two assemblages.

Imagine now that two of the cusps approach very near to the oval, as in fig. 2. I call these two cusps a and b, and I want to attend particularly to two portions of the curve; namely, (1) the branch of the tricusp which is between these two cusps, and (2) the portion of the oval which is between the two points to which the cusps are very near. Suppose these two portions to become flatter and flatter, and to approach nearer and nearer to each other; what becomes of the tangents to them? All these tangents get to differ less and less from the line joining the two cusps. At last, suppose that they all coincide with it, and let us watch what becomes in this case of the curve, its points, and its tangents. First, for the curve; the two other branches of the tricusp join on to the remaining portion of the oval and form a figure like a cardioid, represented in fig. 3. The assemblage of tangents must remain of the third class; it consists just of the tangents to this cardioidal curve, *among which the line ab counts for two.* Lastly, for the points; these are in the first place the points of the cardioidal curve afore-mentioned, clearly enough. But besides these we have to account for the points on the two portions which coincided with the segment ab. These obviously pass continuously into the points upon the linear segment ab, and each of these counts for two. Moreover, there were two series of invisible points of the curve very near to all the rest of the line ab, outside this segment; and at the instant that the two visible portions united on the segment, these invisible portions started into

* Mr Cotterill is, I believe, the first person that ever *saw* a curve of the third class.

visible existence by uniting all along the rest of the straight line. I affirm this dogmatically, and there is no reason for you to believe it, unless you are acquainted with the theory of invisible points. Our assemblage of points which was of the sixth order has then broken up into the points on a cardioidal curve of the fourth order, and the points on a straight line each counting for two.

Other changes have taken place at the same time. I said that the original curve of the third class and sixth order had nine cusps, three visible and six invisible. In the transformation which the curve has undergone, we have seen it acquire a double tangent and lose two visible cusps. Now I assert—dogmatically as before—that besides these two visible cusps it has also lost four of the invisible ones; so that the cardioidal curve which is left has three cusps, one visible and two invisible. And it is a general rule (discovered by Plücker and explained in Dr Salmon's *Higher Plane Curves*, p. 73) that every curve of given class which acquires a double tangent loses thereby six cusps; exactly as a curve of given order which acquires a node loses thereby six inflexions. Of course it is now natural to ask "can we not give the curve a double tangent in such a way as to cut off all the invisible cusps, and leave only the three real ones?" We can do this; but it is necessary first to enter into explanations in regard to two possible kinds of double tangents. Precisely as a double point may be either a point at which two visible branches of a curve cross, or else a conjugate point, the limit of a very small oval; so a double tangent may either have two visible points of contact (as in fig. 3), or two invisible ones*. In the latter case it is called an *ideal* tangent (Poncelet) and appears to have—like a conjugate point—nothing to do with the curve. Now a curve of the third order (fig. 4) may have a double point given to it in two ways. Either we may visibly join together the oval and the sinuous part, making a loop (fig. 5); or we may let the oval shrink up into a conjugate point, at which two invisible

* Salmon's *Higher Plane Curves*, pp. 34, 35. I learn with great satisfaction that already the new edition of this is partly in print. [Published Jan., 1873. The work is now, August, 1879, in a third edition.]

branches of the curve cross each other (fig. 6). Precisely analogous distinctions hold between the two ways in which a curve of the third class (fig. 7) may acquire a double tangent. (Fig. 7 is the same curve as fig. 1, but it has been projected so that the line at infinity cuts the oval part, which consequently resembles a hyperbola instead of an ellipse.) We may either make this go through the process already described, by which it becomes fig. 8, acquiring a real double tangent with visible points of contact; or we may make the angle between the asymptotes larger and larger, till the two branches of the hyperbolic part coincide into a doubled straight line (fig. 9), which is then an ideal double tangent having invisible points of contact. And this kind of double tangent does what was wanted; viz. it removes six invisible cusps, leaving three visible ones.

We establish then two different kinds of curves of the third class having a double tangent. First, there is the cardioidal curve, having visible points of contact with its double tangent, one visible and two invisible cusps. Secondly, there is the simple tricusp, having invisible points of contact with a real double tangent, and three visible cusps. My main business is with the first of these kinds; but before coming to it, I shall make some remarks about two special forms, one of each kind, and mention also some apparently different general forms of curves of the third class not having a double tangent.

Of the cardioidal form the cardioid itself is a particular case; viz. it is a curve of the third class and fourth order with one visible and two invisible cusps. In general, a curve of the fourth order which has cusps at the two invisible points at infinity through which all circles pass is called a Cartesian oval. A Cartesian oval may also be defined as the locus of a point whose distances (ρ, ρ') from two fixed points satisfy an equation of the form $m\rho + n\rho' = c$. The curve has three foci, all in the same straight line, and any two of them may be taken for the fixed points. A Cartesian oval with an additional cusp (this is necessarily real) is a cardioid; the three foci coincide at the cusp. Any curve of our first kind then may be regarded as the shadow of a cardioid; and in order to project such a curve into a cardioid, it is only necessary to project the two invisible cusps into the circular points at infinity*.

A particular case of the other form is the hypocycloid of three branches. This curve has for its double tangent the line at infinity, and touches it at the circular points. Every curve of the third class with an ideal double tangent may thus be regarded as the shadow of a hypocycloid; and in order to project it into a hypocycloid, it is only necessary to project the invisible points of contact into the circular points at infinity.

This curve is the envelope of the asymptotes of all the rectangular hyperbolas that pass through three fixed points. (Steiner). To prove this, it is necessary to shew first that the envelope is of the third class, *i.e.* that three such asymptotes can be drawn through an arbitrary point; and secondly, that the line infinity counts twice as an asymptote, or is a double tangent to the envelope, its points of contact being the two circular points. Now a rectangular hyperbola is one which cuts the line infinity in two points which make with the circular points a harmonic range. If the hyperbola *touch* the line at infinity (become a parabola), its two points of intersection coincide; and they can only do this at one or other of the circular points. There are therefore these two cases in which the line infinity is itself an asymptote; and if we consider a hyperbola very near to one of these cases, we see that the point at which the line infinity is met by the consecutive asymptote is the circular point itself. We have therefore established the *second* of our two facts; that the line infinity is a double tangent to the envelope, [and that the] points of contact are the circular points. To find now the *class* of the envelope, let us enquire how many tangents can be drawn to it through an

* On Cartesian Ovals see Crofton, *Proceedings of the London Mathematical Society* [Vol. VI. pp. 5—18]. On the Cardioid, Purkiss, *Messenger of Mathematics*, [Vol. II. pp. 241—249], and in especial relation to the present theory, Siebeck, *Ueber die Erzeugung der Curven* 3ter *Klasse und* 4ter *Ordnung durch Bewegung eines Punktes*, Crelle, Vol. LXVI. p. 344 (1866).

arbitrary point at infinity. There is in the first place the line infinity itself, counting for two. Besides this, there is the asymptote of the *one* hyperbola of the series that can be drawn through the given point; in all *three* tangents, or the envelope is of the third class. But every curve of the third class touching the line infinity at the circular points is a three-cusped hypocycloid: therefore, &c.*

I said that *in general* a curve of the third class consists of a tricusp surrounded by an oval. But an oval is a thing of the nature of a conic section, and may at times be wholly invisible, like the conic section $x^2 + y^2 + a^2 = 0$. We have accordingly a variety in which the oval has disappeared, and the curve is represented by a tricusp only. The tricusp, however, need not be finite; I have represented in fig. 10 a curve met by every straight line in at least two real points, which cannot therefore be projected into any finite form. To satisfy yourself that it is really a tricusp very similar to fig. 9, draw on a sphere its curve of intersection with a cone, whose vertex is the centre, standing on this curve; it will consist of two equal and opposite tricusps, each with two branches longer than half a great circle.

II.

The Focus of a Double Parabola.

I shall take the liberty of using the name *Double Parabola* to denote a curve of the third class, having the line infinity for a double tangent; the points of contact being any two points whatever at infinity. The reason of the name is tolerably obvious; for the curve has two pair of parabolic branches, and may be derived from the *ensemble* of two parabolas by continuous modification. If the two points of contact are visible, the curve is a central projection of a car-

* On the three-cusped hypocycloid, see a series of papers in the *Educational Times*, Reprint, Vols. III., IV.; and a most elegant synthetic discussion by Cremona, *Sur l'hypocycloide à trois rebroussements*, which appeared about the same time in Crelle, Vol. LXIV. p. 101 (1865).

dioid, obtained by projecting the double tangent to infinity; if invisible, the curve is an orthogonal projection of a hypocycloid. The two kinds may be distinguished as *hyperbolic* and *elliptic* respectively—names on which more light will be thrown in the sequel. The hypocycloid of three branches must then be regarded, in accordance with this nomenclature, as a *circular* double parabola.

Now a *focus* of a curve is a point such that the two lines joining it to the circular points at infinity both touch the curve. Accordingly, a double parabola has in general one focus and no more than one; for the curve is of the third class, and consequently only one tangent can be drawn from one of the circular points, besides the line infinity which counts for two. This single focus lies inside the hyperbolic curve, and outside the elliptic one, moving further away from the curve as it approaches the circular form, the focus of which is away at infinity in no particular direction.

I am going to seek for a geometrical property of this focus, which may serve as a rule for constructing the curve. To this end I form as follows the tangential equation of the curve. Let $f=0$ be the tangential equation of the focus, $i=0$, $j=0$, the equations of the circular points at infinity; $a=0$, $b=0$ the equations of the points of contact with the line at infinity, and $c=0$ the point at infinity on the remaining tangent from the focus to the curve. Then I say that the equation is

$$abf = \lambda \, . \, ijc \text{ ....(1)},$$

where λ is a numerical constant. For the equation is of the third degree, representing therefore a curve of the third class; it is satisfied by the coordinates of the lines fi, fj, fc, shewing that fi, fj are tangents to the curve, and f consequently a focus, and that fc is the remaining focal tangent; and it shews that the three tangents from each of the points a, b (viz. ai aj, ac; bi, bj, bc) coincide with the line at infinity—the points a, b, c, i, j being all on that line—so that a and b must be points on the curve at which that line touches it, unless the line were a triple tangent, which is impossible for a curve of the third class.

To make this equation yield us a relation of lengths and angles serving for geometrical construction, let us take lines A, B, C at a finite distance passing through the points a, b, c respectively, and denote by the letter O the line at infinity. The points a, b, c may now be denoted by AO, BO, CO; viz. a is the intersection of the lines A, O and so on. Let X be a variable tangent to the curve; then substituting its coordinates in the equation (1) thus modified, we have

$$XAO\,.\,XBO\,.\,Xf = \lambda\,.\,Xi\,.\,Xj\,.\,XCO \;\ldots\ldots\ldots\; (2),$$

where XAO means the determinant formed with the coordinates of the lines X, A, O, which vanishes when they meet in a point; and Xf means the result of substituting the coordinates of the line X in the equation of the point f, or (which is the same thing) the result of substituting the coordinates of the point f in the equation of the line X. Now if we observe that*

$$\sin(X, A) = \frac{XAO}{\sqrt{(Xi\,.\,Xj\,.\,Ai\,.\,Aj)}},$$

$$\text{perpendicular from } f \text{ on } X = \frac{Xf}{\sqrt{(Xi\,.\,Xj)}\,.\,Of},$$

* These formulæ are justified as follows. The coordinates of the circular points are taken to be

$$\text{of the point } i,\ x : y : 1 = 1 : \iota : 0,$$
$$\text{of the point } j,\ x : y : 1 = 1 : -\iota : 0,$$

and the equation of the line O at infinity is taken to be

$$0\,.\,x + 0\,.\,y + 1 = 0.$$

Then if the equation of X is $lx + my + n = 0$, we have

$$Xi\,.\,Xj = (l + m\iota)(l - m\iota) = l^2 + m^2,$$

and if the equation of A is $l'x + m'y + n' = 0$, then

$$XAO = \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ 0, & 0, & 1 \end{vmatrix} = lm' - l'm,$$

while $Of = 1$, whatever are the coordinates of f. Making these substitutions, the formulæ become

$$\sin(X, A) = \frac{lm' - l'm}{\sqrt{(l^2 + m^2)}\sqrt{(l'^2 + m'^2)}},$$

$$\text{perpendicular from } f \text{ on } X = \frac{lx' + my' + n}{\sqrt{(l^2 + m^2)}} \ (x'y' \text{ coordinates of } f),$$

which are the ordinary ones.

we may transform equation (2) into the following

$$\frac{\text{perpendicular from } f \text{ on } X}{\sin(X,\ C)} = \frac{\mu}{\sin(X,\ A)\sin(X,\ B)} \ldots\ldots (3),$$

where
$$\mu = \frac{\lambda\sqrt{(Ci\,.\,Cj)}}{\sqrt{(Ai\,.\,Aj\,.\,Bi\,.\,Bj)}\,.\,Of}.$$

Now C was any line through the point c, that is, any line parallel to the focal tangent; but I shall now regard it as the focal tangent itself. This being so, it is clear that $\frac{\text{perpendicular from } f \text{ on } X}{\sin(X,\ C)}$ = distance from f to X, measured along the focal tangent; or it is the segment on the focal tangent determined by the variable tangent X. We have arrived then at the theorem that *this distance is inversely proportional to the product of the sines of the angles which the tangent makes with the asymptotic directions* A *and* B. Let δ be the distance from the focus to the point of contact of the focal tangent; then taking X to be the focal tangent itself, we get

$$\delta = \frac{\mu}{\sin(A,\ C)\sin(B,\ C)},$$

and, eliminating μ, we may state our theorem in the form

$$\frac{\text{segment on focal tangent}}{\delta} = \frac{\sin(A,\ C)\sin(B,\ C)}{\sin(A,X)\sin(B,X)},$$

= anharmonic ratio of points a, b, c, XO.

We may simplify still further this enunciation by remembering that any squared diameter of a conic is inversely proportional to the product of the sines of the angles which it makes with the asymptotes, and at the same time directly proportional to the parallel focal chord. Finally, we arrive at the following

Construction. On any fixed line C through a focus f of a conic section let a distance fp be measured equal to a focal chord, and through p let a line X be drawn parallel to the focal chord; the envelope of the line X as the chord turns round will be a double parabola which has infinite branches parallel to the asymptotes of the conic, the point f for focus, and the line C for focal tangent.

According as the conic is ellipse or hyperbola, the double parabola will be elliptic or hyperbolic. If the conic is a parabola, it assumes an intermediate form, the semi-cubical parabola $ay^2 = x^3$. If the conic is a circle, the envelope reduces to a point, as it ought to, being a hypocycloid whose focus is at an infinite distance compared with the dimensions of the figure.

III.

THEOREM. *The locus of the foci of all the double parabolas which touch five fixed lines is a circle.*

A double parabola is uniquely determined by six tangents. For a curve of the third class is determined in general by nine tangents; and to be given that a particular line is a double tangent is equivalent to three linear tangential conditions. The double parabolas touching five fixed lines are therefore a singly infinite series like the conics inscribed in a quadrilateral, and there is only one of them that touches another arbitrary line. Let an arbitrary line be drawn through the circular point i; there is then one double parabola of the series that touches this line. From the point j one tangent only can be drawn to the curve, for it is of the third class, and the line infinity already counts for two tangents. When therefore the tangent from i to the curve is given, the tangent from j is uniquely determined; and so likewise when the tangent from j is given, the tangent from i is uniquely determined. These two tangents are therefore corresponding rays of two homographic pencils, and the locus of their intersection (*i.e.* of the focus f) is consequently a conic section passing through the points i, j, that is to say, a circle.

COR. 1. The foci of the five parabolas which touch every four of the five lines are on this circle.

For a double parabola, being a curve of the third class, may break up into a conic and a point; namely, into an ordinary parabola and a point at infinity. Among the double parabolas which touch the five lines are to be reckoned five such degenerate cases, consisting of the point at infinity on one of the lines and the conic parabola which touches the other four. The focus of this degenerate form is (as is obvious from the definition of a focus) simply the focus of the conic parabola; whence the corollary follows, and Miquel's theorem is proved.

COR. 2. If we take six lines to start with, we may in this way determine six circles, omitting the lines one by one. These six circles all meet in a point, the focus of the double parabola which touches the six lines.

The transformation (as to its tangents) of a cardioidal curve into a conic and a point is illustrated by fig. 11, which represents such a curve very nearly consisting (as to its visible points) of a conic and a doubled finite portion of one of its tangents.

IV.

DEVELOPMENTS.

So far we have considered the following series of propositions:

(1). Given three lines, a circle may be drawn through their intersections. (Euc. IV. 5.)

(1′). Given four lines, the four circles so determined meet in a point. (Well known.)

(2). Given five lines, the five points so found lie on a circle. (Miquel.)

(2′). Given six lines, the six circles so determined meet in a point. (Sect. III., Cor. 2.)

I shall now shew that the series is interminable; that is, that $2n$ lines determine $2n$ circles all meeting in a point, and that for $2n+1$ lines the $2n+1$ points so found lie on the same circle.

Connected with Prop. 1, however, there is another theorem which is susceptible of generalization. If from any point in the circumscribing circle we draw perpendiculars to the sides of a triangle, the feet of these perpendiculars are in one straight line. I call this ($1p$); the corresponding pendant to Miquel's theorem is,

($2p$). If from any point p on the circle in Prop. (2) we draw perpendiculars on the five lines, their feet lie on a conic passing through p.

So again we have

(3). Given seven lines, the seven points obtained as in (2′), are all in one circle.

($3p$). If from any point p on this circle we draw perpendiculars on the seven lines, their feet will lie on a cubic having a node at p.

And generally

(np). If from any point p on the circle determined by $2n+1$ lines we draw perpendiculars to them, the feet of the perpendiculars will lie on a curve of order n passing $n-1$ times through p.

To prove these results, it is necessary to consider a curve of class $n+1$, touching the line infinity n times. I shall call such a curve an n-fold parabola. It is in fact of order $2n$, and has n pairs of parabolic branches*. From any point at infinity

* The tangential equation to an n-fold parabola is always

$$fa_1a_2 \dots a_n = \lambda . ijc_1c_2 \dots c_{n-1},$$

where $a_1, a_2, \dots$ are points of contact with the line infinity, and $c_1, c_2, \dots$ points at infinity on tangents from the focus to the curve. From this we may deduce at once a construction analogous to (3), p. 49. In the case of the triple parabola this construction takes the simplified form

$$\text{intercept on } X \text{ made by } A_1, A_2 = \frac{\mu}{\sin(X, C_1)\sin(X, C_2)\sin(X, C_3)},$$

where A_1, A_2 are the focal tangents, and C_1, C_2, C_3 the asymptotic directions. μ may be determined by making X coincide with A_1, A_2 successively.

not a point of contact, there may be drawn one other tangent to the curve; the line infinity counting for n, and the class being $n+1$. Such a curve therefore has always one and only one focus. Now a curve of class $n+1$ is in general determined by $\frac{1}{2}(n+2)(n+3)-1$ or $\frac{1}{2}(n^2+5n+4)$ tangents; but an n-fold tangent of given position is equivalent to $\frac{1}{2}n(n+1)$ single tangents in its effect upon the determination of the curve. The number of tangents finite in position which determine an n-fold parabola is the difference of these numbers, $2(n+1)$. All the n-fold parabolas, therefore, which touch $2n+1$ fixed lines form a singly infinite series; and it is easy to see that the locus of their foci is a circle. For if we draw an arbitrary line through the circular point i, *one* curve of the series can be drawn to touch it, and this determines uniquely the tangent from the other point j. These two tangents then, as before, are corresponding rays of two homographic pencils, and their intersection must trace out a conic through the points i, j, that is to say, a circle.

Now among these n-fold parabolas are included $2n+1$ degenerate cases, each consisting of an $(n-1)$ fold parabola and a point, viz. the point at infinity on one of the lines, and the $(n-1)$ fold parabola determined by the other $2n$. The foci of these are therefore points on the circle in question, and we may enunciate the following propositions:

(n). Given $2n+1$ lines, the foci of the $2n+1$ $(n-1)$ fold parabolas each of which touches $2n$ of the lines are on the same circle.

This circle is the locus of the foci of n-fold parabolas touching the lines, and therefore

(n'). Given $2n+2$ lines, the $2n+2$ circles so determined meet in a point, the focus of the n-fold parabola touching the lines.

By successive applications of this theorem with suitable values of n, we are able to see that the series of statements at the beginning of this section may be continued indefinitely.

Let us now inquire what is the *pedal* of an n-fold parabola with regard to the focus, that is to say, what is the locus of the